Growth and Fiscal Policy in Dynamic Optimising Models

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Abstract

This PhD thesis considers the dynamics of fiscal policy in a two-country world when growth is driven by the accumulation of private capital and public infrastructure. I study permanent growth differentials, the dynamics of optimal and time-consistent policies, the issue of policy coordination, as well as the accumulation of debt.

 I am grateful to Paul Levine for helpful discussions at all stages of this work. He also suggested the title.

Pour Hexe

1 Introduction

One very important, if not the central debate of economics in general is the comparison between uncoordinated outcomes (often called equilibria) and coordinated ones (usually called optima). The idea that underlies that distinction is simple. In any social situation, one outcome arises if the situation is simply allowed to develop by itself. But if there is an intervention by some coordinating authority, welfare may improve; in fact it may also deteriorate, or the welfare of some agents may be improved at the expense of the welfare of others. Economists often attach little interest to the institutional aspects of coordination, in particularly when looking at coordination between autonomous agents. An example for that would be the coordination between governments on the international stage. However within each economy, the debate on the extent of cooperation becomes more focused around the extent and direction of state intervention. This has been one of the central subjects in the economic debate. The only point that everybody agrees on is that in an economy not everything can be done by the private hand.

My work starts from that simple point. I aggregate all measures that the state undertakes in the economy under the term "fiscal policy". This implies a broad view. I study fiscal policy in terms of broad aggregates, an aggregate for government consumption, one for government investment and one for taxation. I also take a broad view of the timing, and emphasise the long-run implications of policy. However this does not come at the expense of analytical rigour in each time period. Continuing the simple aggregative framework, I consider two agents; the private sector and the public sector (government). Both are optimising intertemporal objectives. One central feature of the thesis is to analyse fiscal policy in the context of these intertemporally optimising agents.

The private sector lives forever, it owns some capital stock both at home and abroad. Its derives income both from selling labour to firms and from the ownership of capital. It pays taxes on that income. In every period it will have to decide on how much to consume and save. The capital stock and national output are thought of as the same commodity, therefore capital could be consumed at any period, i.e., there is no positivity constraint on investment. The private sector is only constrained by its intertemporal budget constraint. That constraint says that the present value of all future expenditure must be smaller than the present value of all future post-tax income streams. Since the private sector pays taxes, it will have to make a forecast over the extent and the timing of income taxes before it makes its consumption/savings decision. This is not a trivial task.

The public sector is an agent that provides both a public consumption commodity and public investment. In principle there is only one physical commodity in each country. However when the state obtains some of that commodity through the collection of taxation, then there appears a second "commodity" that directly enters in the felicity function of the private sector representative agent. Similarly, when the state makes public investment available then there is an externality impact on the productivity of firms. Thus the government has a consumption/investment decision to solve just like the private sector has. The complication lies in the interaction between the two agents, both in the short run and in the long run.

In the long run, I assume that the economy reaches a steady state where all aggregates grow at a constant rate. The long-run growth rate in the economy is not fixed by technological factors but depends on the actions of both the private and the public sector, i.e. there is "endogenous" growth. To generate persistent growth, I assume an externality of a publicly provided stock called infrastructure on the productivity of private capital. Using constant returns to scale assumptions, national output becomes a function of the private and public capital stock only. Since national output then only depends on accumulable factor, endogenous growth is possible. The crucial aspect of this type of endogenous growth model is that the government has an important impact on the long run behaviour of the economy. Like the private sector, the government faces a problem of consumption vs. savings. It can either spend on government consumption in the current period, thereby augmenting current felicity or spend on infrastructure, which increases the amount of resources in the next period and raises growth.

How large that rate of growth is is of key importance to welfare, but it is by no means the only welfare criterion. The fraction taken up by consumption in national income is of similar importance. It is trivial to imagine a society that is fast-growing, but where the share of consumption is small. That society may in fact fare worse in a steady state than another economy that grows less fast but where the share of consumption in national income is higher. Note that the welfare criterion here is simply that of a household born on any period in the steady state and living throughout her infinite life in that steady state.

The long run of endogenous growth models has been studies extensively. Many papers study only the long run. They examine comparative statics of policy, i.e. they compare steady states with various values of the policy variable. But it is important to note that the criterion of choice between steady states is unrealistic, because it implies an arbitrary choice between steady states. In some very simple systems, the economy is always in a balanced growth path. That is, for example, the case if all stocks depreciate fully during the production period. In that case the problem faced in each period is identical, the initial capital stock is only a scale parameter. But as soon as there are more than two stocks involved in production that do not depreciate fully, even if the aim of the consumer would be to get as quickly as possible to the steady state, there would necessarily be some transitory dynamics from the initial position to the steady state. This raises two problems

- Because of the complicated non-linear relationship between consumption, investment and growth, there is no analytical solution to the problem in most settings.

- If the private sector reacts to policy a problem of time inconsistency is likely to occur. This is a difficult conceptual and technical problem.

Because of the first problem there are two paths that are taken in the majority of the literature. One is to look at the welfare comparison between steady states and ignore the transitional dynamics, or depart directly from a framework where transitory dynamics do not exist. Another approach is to analyse the effects of shocks that hit a steady state without the analysis of welfare maximisation. The second problem is usually assumed away, by considering that all future policy instruments can be set at the current period.

This thesis deals with both problems at the same time. There is off-steady-state maximisation and appropriate treatment of the time-inconsistency issue. It is the only piece of work in this that solves for a time-consistent fiscal policy in the presence of both transitory dynamics and forward-looking private sector behaviour in the absence of constraints on the usage of instruments.

That dazzling technical virtuosity apart, is there anything else that makes this piece worth reading? There are several more basic issues that the thesis solves or touches on. The most important simple issue is the question of growth divergence. Here I have a simple question, and a simple answer. The simple question is "Is it possible for one country to perpetually grow faster than another with which it trades?". Astonishingly the answer to that question is "Yes it is, under certain conditions". In the case where each economy is specialised in the production of a tradeable output, I show that an appropriate adjustment of the terms of trade in each period allows for a steady state where each country grows at different rates. But having shown that, I also examine a permanent asymmetric shock to a symmetric steady state that leads to growth divergence. For a linear approximation, I show that the utility in one country becomes arbitrarily low as time progresses, because consumption is reduced following a steady loss of foreign assets. This means essentially that although it is possible for countries to grow at different rates, we can not determine a linear path of adjustment that would lead from one steady state to another.

The thesis then examines the optimal and time-consistent policies. When I try to explain this to fellow economists, they usually are baffled by the idea of a time-inconsistency problem in a fiscal policy model. They are used to thinking about a time-inconsistency problem as a feature of monetary policy. In fact the problem is far more general than that. In any model where the private sector's action depends on future government policy, the issue of time-inconsistency arises, unless the model is such that a decision on policy is only taken once. Most economists discuss the time-inconsistency problem for monetary policy issues only. This is essentially a repeated static situation, because the initial money stock one does start of with does not have a rôle in the model in the sense that it could be normalised to unity at the start of each period. Under those circumstances, the game starts anew each period, and the time-consistent solution involves a penalty in each period. A mechanism that enforces the time-consistent policy will bring a welfare gain for all participants in such a situation. None of the restrictions of this simple monetary game apply here. First, since the government's decision in the previous period does affect the productive capability in the economy in the current period, the decision to play time-consistent or time-inconsistent has repercussions not only on the current period but also on all future periods. For any fixed set of actions to be taken by all participants from the current period, different initial conditions will lead to different welfare.

Second it is not true that the time-consistent policy involves a loss in each period. In fact in the long run the time-consistent policy is likely to lead to higher welfare. This is one of the central results of this thesis. To understand this on an informal level, we can distinguish two scenarios.

Imagine first an optimal, time-inconsistent policy that involves "indulgence" initially, and "sacrifice" in the future. What would does the time-consistent policy look like? Since the reversal from indulgence to sacrifice would be not be time-consistent, the tendency to indulge would continue during all periods, but of course since past indulgence has eroded the possibility to indulge in the current period it is likely that as we move to the future, we indulge less in every period, because current indulgence reduces the possibility for future indulgence. In the long run we run down our opportunities to indulge to zero. Clearly the long-run welfare in such a system will be poor. Not only will the time-inconsistent solution be better from the initial point onwards, it will also do better in the long run.

But now assume that the opposite is true, that the optimal policy consists in making a sacrifice in the early periods and allow for indulgence in the later periods. Again, the policy reversal is time inconsistent. In the time-consistent solution there will be a sacrifice in each period, but in the later period, the sacrifice will bring fruit and allow for higher consumption possibilities. In that case the time-consistent policy brings higher welfare in the longer run than in the short run and there is a risk that the time-consistent policy outperforms the optimal policy in long run welfare.

Are most economic optimisation problems rather leading to trajectories of the "indulge then sacrifice" type or the "sacrifice then indulge" type? I am not aware of any broad study of this question, but it seems to me that the latter type is much more prominent than the former. The latter situation arises for instance in models where there is capital accumulation problem and initial capital falls short of an overaccumulation level. This is the situation relevant for all models in this thesis. It is also typically true in many models where the government can issue debt or assets and where the initial position of the government falls short of avoiding distortionary taxation at some stage in time. This is the situation examined in Section 9.

Before that section on debt, I examine the coordination issue that arises in a two-country model. In a model that is very close to mine, Devereux and Mansoorian (1992) show that the extent of government investment is not subject to an international coordination problem, but that only government consumption is. The spillover effect depends a great deal on the elasticity of substitution in consumption. For some parameters higher growth at home raises growth abroad, for some others higher growth at home lowers growth abroad, and for values that are in the "received wisdom" range the results are ambiguous. Indeed I find that for the particular calibration that I choose, and for time-consistent regimes, results are fairly close and the question of coordination of policy has little relevance for welfare.

Finally the issue of government debt features prominently in the last chapter of the thesis. Government debt is a very difficult topic of the analysis for economic policy based on optimising behaviour where government and private sector share roughly the same objectives. It has been shown time and again that when the government can levy debt, it is not optimal to levy positive debt but instead the government should be a creditor to the private sector. This makes perfect economic sense but it is at odds with what we observe. The problem could be addressed in various ways. One can follow the route of the political business cycle literature, and assume that there is an important gap between private and public sector. A lot of different scenarios have been proposed in this literature. All have the same problem of non-robustness, in the sense that if the game is slightly changed the outcome is affected in an important way. Another avenue towards addressing the problem is to stick with the aggregative framework but to assume that there is a parametric difference between the private and public sector's preferences. Usually it is assumed that the public sector has a higher discount rate, for example a government is facing elections. It has been shown that this type of assumption can generate debt but only if one is prepared to make extreme assumptions about the difference between private and public sector discount rates. Another approach is to give up the representative agent framework for the private sector. There have been recent efforts in this approach but this literature is to young too allow for meaningful conclusions on the debt front.

In this thesis, I do not address this problem, instead I am more concerned about the details of implementing solvency. A government is solvent if the present value of its long-run assets is positive. In any period it is not clear how the future measures that close the gap between expenditure and income in the long run can be enforced today. This is not only a technical problem. I believe that this is in fact the most important conceptual issue facing fiscal policy when debt can be levied in an infinite horizon model. There is always another period when taxes can be raised and/or expenditure lowered, thus the introduction of a constraint on action in the current period is difficult. One idea that enforces solvency (without modeling it) is to directly penalise debt. That is not a very satisfactory solution since the size of the penalty is arbitrary and its time structure may induce the government to shift resources to avoid the penalty rather than for reasons that related to the primary objectives of policy. In this thesis I set out to introduce solvency directly, by treating the value of the current government debt as non-predetermined and set it equal to value that it would have under the solvency constraint. This novel concept of government debt leads to convincing model properties and results.

To summarize: this thesis studies the issues of growth asymmetry, time-consistent policies and policy coordination in a two-country endogenous growth model with government capital, with and without government debt. I first review the literature on these topics in Section 2. I set out the general model in Section 3. In Section 4, I address the first issue of the paper, the existence of a steady-state equilibrium where both countries' GDP grows at different rates. In Section 5, I seek an appropriate linear-quadratic approximation for the model. This approximation is used in three simulation exercises. Section 6 examines the dynamic reaction of the model to shocks that lead to asymmetric growth. Section 7 contains the results for optimal and time-consistent policies in a single country model. Section 8 examines cooperation vs. non-cooperative policies that are time-consistent. Section 9 introduces the issue of government debt. Section 10 concludes.

The thesis is my work only. However the solution procedures for the optimal and time-consistent solutions are largely based on Currie and Levine (1994). They are presented in Appendix A and Appendix B.

2 Literature review

In this Section I review some of the literature relevant to the subject matter. In Subsection 2.1 I look at the time-inconsistency literature. In Subsection 2.2 I review the literature on simple models of endogenous growth with public infrastructure. In Subsection 2.3 I examine what could be called the growth differential literature (if it existed). Subsection 2.4 considers the literature on debt. In the last Subsection I look more closely at the model whose formal structure comes closest to mine.

2.1 The literature on time inconsistency

This thesis studies fiscal policy under a time-consistency constraint. As its name suggests, time inconsistency arises in dynamic models. Roughly, when an agent makes a plan over several periods, there is an incentive to deviate from the earlier plan in the later periods. Therefore if there is no mechanism by which an agent can be forced to comply with her earlier plans, the outcome over the whole period is suboptimal since it does no longer correspond to the initial optimal trajectory.

In issues of economic policy, the most frequently studied problem is the one of a government that faces a forward-looking private sector. If the government is able to precommit to any path for the variables under its control, then the government adopts a standard decision-making problem involving an intertemporal objective. In that case it can immediately influence the private sector's intertemporal behaviour. Otherwise the government will be reoptimising in each period. In that case the impact of the policy on the private sector will be more limited, because the private sector will expect the government to reoptimise.

Academic economists have long been worried about time inconsistency, to the extent that a majority seems to perceive time inconsistency as a problem that begs a solution. One important reason for this view is that students of economics are familiarized with the issue within the framework of stabilisation policy. A typical example is the following story. There is a government that would like to stimulate national output, but keep inflation low. It operates in an environment where only surprise inflation can increase output. The public formulates rational expectations. Therefore the optimal outcome is not to inflate; however that outcome is not time-consistent. If the public were to believe in the promise of low inflation then the government would have an incentive to create surprise inflation. Thus time inconsistency leads to a dead-weight loss because society ends up in an equilibrium with costly inflation that is anticipated, and therefore yields no output gain. When we repeat that game in this type of model the time inconsistency of the no-inflation decision makes for a loss in each period, no matter what state the economy is in when the period starts. Within this particular framework it makes sense to look for "solutions" to the "problem". To date, the search has not been easy. None of the proposed solutions has won unanimous support.

An influential idea has been the trigger-strategy equilibrium by Barro and Gordon (1983). In this setting a homogeneous private sector is assumed to adopt the strategy to believe that the government will stick to the time inconsistent behaviour until it has been found out that it did not do so on one single occasion. Once the government's reputation is ruined in this way, the public will expect the government to behave in the time-consistent way in all periods. Thus a government facing a public that has adopted such a strategy will, depending on the magnitude of its discount factor, prefer to pick the time-inconsistent policy rather than deviate from it. Thus the time-consistency "problem" is "solved".

It is easy to condemn this approach as oversimplistic. The most important problem is that the trigger strategy that this private sector is following is simply exogenous. In a model with no uncertainty, a strategy to assume no deviation from the time-inconsistent behaviour until such a deviation occurs seems reasonable enough. The verification of the government's commitment is easy and imposes no costs. But as soon as the government's grip on its policy targets is imperfect, there are immediate problems of state verification. All the private sector can do is to calculate an ex post probability of the government's intent to deviate from the time-inconsistent solution. The problem becomes even more complicated when we assume that the public consists of decentralised agents. These may agree that a trigger strategy is the right thing to do in principle, but at what point they should assume that the government has been deviating deliberately from the time inconsistent behaviour? And if they think that the government did not stick to the rule it set out, for how long should the punishment period last? These issues seem to be very difficult to coordinate. No paper has found a compelling answers to these questions.

A second approach towards solving the time-inconsistency problem is to set some social contractual obligations as proposed by Kotlikoff, Persson, and Svensson (1988). This is a rather talmundian idea that has sparked little interest. A more seminal approach to the time-consistency problem in the monetary policy area has been the "Conservative Central Banker", introduced by Rogoff (1985). The idea is to appoint somebody with preferences that are very inflation averse as the head of the central bank. The implementation details have given rise to a very large volume of literature, with some schemes developed that are rather complicated. I think it is best to remain silent about this branch of the literature since it is not of much interest to fiscal policy. At least as far as *I* know, nobody has proposed a "conservative treasurer". I would not blame lack of imagination on the part of economists. There would be formidable obstacles to such an idea. The power to levy taxes is a traditional prerogative of parliaments, thus institutional reform would be hard. Second, the objectives of fiscal policy are much more complex than the ones of monetary policy, such that it is much more difficult to legislate rules for the "conservative treasurer".

For fiscal policy issues, the most frequently studied problem has been the financing of an exogenous (possibly stochastic) sequence of government expenditures by distortionary taxation and debt issue. Lucas and Stokey (1983) have suggested that the time-inconsistency problem can be overcome if the government has a sufficiently large set of maturities at which it can issue debt. More precisely any government at the start date, (say government 1) of the economy can find a structure of debt issued at different maturities such that, if subsequent governments (at time $t \geq 1$) honour the payments—interest and principal—of the debt contracted by the government at the start date 1, their optimal choice under that repayment constraint will be to follow the optimal path chosen by government 1. This is an elegant theoretical argument, but its practical implications are limited. First the model is derived in an economy that directly transforms labour into a consumable commodity without the need for capital. Second, under the stochastic version of the model the government would need to issue state-contingent debt i.e. debt claims that would give rise to claims only if a certain expenditure is realized. Third, besides the conventional problem of state verification, there is additional level of complication that would arise if expenditure levels were endogenous, possibly decided by the same agents that issued the contingent claims. Fourth, it is not clear why the subsequent government should be forced to execute the debt commitment. This point was taken up by Chari and Kehoe (1993) who allow for governments to default on the debt in the model of Lucas and Stokey (1983), without uncertainty. Clearly any government that faces a positive debt obligation would refuse to pay it. Therefore a plan that would incorporate some debt could not be supported. They show that earlier papers where positive debt was possible in equilibrium, like Calvo (1988) and Bulow and Rogoff (1989) relied crucially on the assumption that there is a *direct* cost to default other than the loss of reputation. The solution proposed by Lucas and Stokey (1983) relies on an infinite cost of default. Forcing governments not to default requires some external agent, say a constitutional court. Therefore it appears questionable why the constitutional court would not rather force the government at time t to execute the plans of government 1, without the need to compute the complicated asset management strategy.

These limitations of the approach notwithstanding Persson, Persson, and Svensson (1987) have extended the work by Lucas and Stokey (1983) to include monetary policy, but under the severely restrictive assumption that the money stock is the opposite of the discounted sum of government debt of all maturities. Therefore the initial stock is not predetermined. Thus the authors rule out any benefit from surprise inflation. They also do not consider the problem of optimal taxation of a predetermined stock.

A rare attempt to assess the time-consistency problem of fiscal policy empirically is Swadroop (1993). He builds a model with an infinitely-lived consumer endowed with one unit of leisure per period and a stock of government bonds at the end of her life. She has a constant returns-to-scale production function of labour only at her disposal. Government spending is exogenous and stochastic. The government can use a distortionary tax on income or one-period government debt to finance expenditure. The author then computes the Euler conditions associated with a benevolent government and claims that these conditions hold for a time-consistent policy. Unfortunately, that is wrong. He recomputes the time-inconsistent solution in each period, which would imply the government surprising *itself* each period by reoptimising. The decision in the first period must be taken in the knowledge that in the next period the government will solve the same problem again. Despite the fact that the model is not correctly set up, his empirical test are still of some interest since we can interpret it as testing for optimal (time inconsistent) fiscal policy. Using the General Methods of Moments technique (see Hansen and Singleton (1982)) on annual US data from 1937 to 1985 he can not reject the optimal taxation model. However it should be noted that the method is not really suited for testing a hypothesis.

2.2 Endogenous growth models with government infrastructure

The model in this thesis incorporates a private sector and a government. The government may spend on a publicly provided consumption good or on the building up of infrastructure that indirectly enters into the production function of firms. Arrow and Kurz (1970) have such a production function but do not study the occurance of endogenous growth. For permanent endogenous growth to occur, it suffices that in the steady state the government maintains the infrastructure as a constant fraction of GDP. Then the productivity of capital is bounded away from zero and perpetual growth is possible. This idea is pioneered by Barro (1990). He models a single-commodity world where production is a function of a capital stock that does not depreciate and the *flow* of public services. There are three key results to his paper. The first is that the maximisation of welfare is equivalent to the maximisation of the growth rate. Second, the optimal constant tax rate for spending on investment is equal to the share of public investment in national output. And third, when public consumption (exogenous) is taken into account, it is optimal to levy an additional tax to pay for these services and the optimum investment in infrastructure will be unaffected.

These strong conclusions have invited further examination and qualification. Most of these additions have involved changing some of the mechanics of the Barro model. Futagami, Morita, and Shibata (1993) modify a single aspect of the Barro model by modelling government capital as a stock rather than a flow. This considerably improves the realism of the model at the expense of analytical simplicity. The result that growth is maximized when taxation is equal to public investment carries over from Barro to their model, as long as they abstract from government consumption. However welfare maximisation is no longer equal to growth maximisation because we have transitory dynamics. The authors show that if the tax rate is constant, then the steady state is unique and that there is a unique stable path that converges to the steady state. In addition they demonstrate that the optimum tax rate is smaller than the one that maximizes growth, because growth maximisation implies that future consumption streams are discounted at a rate 0 vis-à-vis current consumption. The analytical solutions of the maximisation problem are not addressed because they are too complicated.

Glomm and Ravikumar (1994) present analytical results in the case where both private and public depreciate fully during the period and preferences are logarithmic. Government consumption is absent, but infrastructure spending may exhibit varying degrees of non-rivalry. Each individual firm produces with constant returns to scale capital and labour, but production is premultiplied by a term that depends on government spending divided by a Cobb-Douglas type index of private factor usage. Thus government consumption is a shift parameter in private production but the intensity of private factor usage will limit its impact i.e. public services are subject to a congestion effect. The authors then solve for the dynamic programme of the private sector and study the optimization problem the government. Unfortunately, the quite elegant formulation of congestion has no impact on the optimal growth rate, which is constant and differs from Barro (1990)'s by being premultiplied with the discount factor. The presence of the discount factor can be explained as follows. Barro restricts his policy to time-invariant taxation. In the initial period, capital is predetermined, but infrastructure is not since it is a flow. It is *current* infrastructure that will affect current production. In that case, government spending is allocated to the sole objective of maximizing both current and future output. Glomm and Ravikumar however assume that current production depends on past investment. Therefore output in the first period is predetermined. Increasing taxes today therefore involves a sacrifice in current consumption and the solution becomes dependent on the discount factor. Maximizing welfare, in that scenario, is not equivalent to maximizing growth.

The problem of congestion is also taken up in Barro and Sala-i-Martin (1992). They use a simple model of congestion where, in order to maintain the same aggregate level of government services, the provision of these services has to rise with the level of GDP. They find that when there is no congestion, then lump-sum taxes are compatible with the social optimum, but in the presence of congestion, a proportional tax on output may preserve the social optimum. Futagami and Mino (1995) investigate congestion in the presence of threshold externalities. In a simple formulation they premultiply the Barro (1990) production function by a term that takes one value for up to a critical value of infrastructure, and a higher one beyond that threshold value of infrastructure. Alternatively they assume that that the multiplying term is a logistic function of infrastructure. They find that the resulting paths, even for ad hoc constant tax policies will display multiple equilibria both for the long run and along the trajectory. The realisation of any particular equilibrium is dependent on the private sector's expectations.

Lau (1995) extends Glomm and Ravikumar (1994) to include government consumption. Like theirs his model is in a permanent steady-growth state. Assuming that preferences are logarithmic, he can compute the optimal—from households' preferences view—share of government spending on consumption and investment in GDP. It turns out that the government consumption is lower under welfare than under growth maximisation, and that government consumption is higher. Therefore, assuming that governments are close to the welfare maximizing policy, an increase in government consumption should decrease growth, but an increase in government investment should increase growth, which is what Barro (1991) found in an empirical study.[1]

A paper in a similar vein is Lee (1992). His production per capita is a Cobb-

[1]See Hsieh and Lai (1994) and Lin (1994) for further references to the empirical literature.

Douglas in the private capital stock per head and the aggregate public capital stock. Both stocks do not depreciate. He simultaneously studies government consumption, government investment and lump-sum transfers to private agents. He manages to solve for the optimum policy of the government when it acts as a leader over the private sector. He finds that there are two local optima, one with a slow growth rate, high government consumption and high transfers and distortionary taxes, and the other with low taxes, high government investment and low transfers. The conclusion that the government should finance positive lump-sum transfers in the first equilibrium using discretionary taxation appears odd. His results should be taken with caution. There appear computational mistakes in the displayed equations after his equation (10) and after his equation (13).

The effect of fiscal policy in an endogenous growth model has also been examined by Turnovsky and Fisher (1995). The basic production framework is the same as Glomm and Ravikumar's, but no specific functional form is assumed. An additional level of generality is added by assuming that labour supply is elastic. The government finances consumption and infrastructure expenditure through lump-sum taxation. The authors are interested in the effects of permanent and temporary changes in fiscal policy. First, when there is an increase in government consumption spending, the increase in taxation needed to finance it will reduce private sector income and consumption. The marginal utility of income increases, therefore households will increase their labour supply. The increase in labour supply raises the productivity of capital and results in additional capital accumulation, potentially leading to a *rise* in the growth rate. In addition to the effect of taxation, there is a direct effect—through the representative agent's felicity—of government consumption on the marginal rate of substitution between consumption and leisure. This effect could potentially reverse the adverse effect of taxation on the representative consumer's utility. Increased expenditure on infrastructure will have the same taxation effect since it also needs to be financed by tax. In addition, an increase in public infrastructure results in an increase in income that will tend to counter the taxation effect. The total impact of increasing infrastructure on the private capital stock is therefore ambiguous. The authors then show that when technology is Cobb-Douglas, an increase in government consumption will increase the private capital stock by more than an increase in public infrastructure would. However the welfare effect and growth effects of raising one or the other are ambiguous.

The interaction between public expenditure and labour supply decisions are also taken up by Devereux and Love (1995). They show that government spending can have an impact on growth even in the absence of direct government investment into the capital stock. Their model comprises physical capital and human capital. Labour supply is elastic, and human capital accumulation is not taxed. When a permanent increase in government spending occurs, its effect will depend on how the increase is financed. When the government uses a lump-sum tax, the private sector wealth is reduced. Both leisure and consumption are normal goods; therefore there will be a drop in private consumption and an increase in the labour supply. In equilibrium, the rate of return on accumulating human capital increases, and so does the rate of return on physical capital. Therefore the growth rate will rise. This result, which is very close to Turnovsky and Fisher hinges on the lump-sum taxation assumption. When the lump-sum tax is replaced by an income tax, Devereux and Love show that a permanent increase in taxation will reduce growth via a reduction in the private capital stock.

A comprehensive recent study of fiscal policy with optimising government spending is the "Model 3" of Jones, Manuelli, and Rossi (1993). Contrary to Turnovsky and Fisher (1995), they study distortionary taxation. There is also an interesting variation on the stock/flow specification of the productive input, where the investment is homogeneous function of degree one in private and public gross fixed capital investment[2]. In addition, there is an important feature that is absent in the previous contributions: the government's budget constraint is relaxed to its intertemporal version, i.e. the government may accumulate debt or assets. The government would like to use lump-sum taxation, and even if there is no lump-sum taxation is available, the government can tax the current capital stock. Since this capital stock is predetermined, taxing it mimics a lump-sum tax. Therefore the optimum solution consists in taxing the existing capital stock heavily in the first periods, until a surplus is built up that allows to finance future commitments without the necessity to levy further distortionary taxes. There are two problems with that solution. The first is that the authors need to impose a restriction on the tax rate to prevent it to hit over 100%. The computed trajectory then depends heavily on the restriction that is adopted, in fact when control is implemented, the tax rate jumps to the bound and remains there for many periods. Therefore the bound drives the solution. The second problem is that that solution is not time-consistent. At any point in the future, as long as there is revenue to raise, there remains the temptation to raise taxes again. The authors acknowledge that "This is clearly a problem with the solutions presented in connection with these models" (p. 511) and that " ... a more complete treatment of the problem including these issues would be of considerable interest" (p. 487).

This issue is addressed by Krichel and Levine (1995). Here the authors build a model with overlapping generations à la Yaari (1965)–Blanchard (1985). There are two groups of consumers. The first group maximises a discounted sum of logarithmic felicity from consumption of a private and a publicly provided consumption commodities. The second group are liquidity constrained and spend the current income on the private consumption commodity. The production side of the model is similar to Barro (1990), but both private capital and infrastructure are modelled as stocks. This precludes an analytical solution of the off-steady-state behaviour. The government can finance expenditure by levying a distortionary income tax or through issuing debt. To simplify the problem the simulation exercise leaves total government consumption and infrastructure (as fractions of GDP) at some exogenous calibrated value. The only decision is the financing of the expenditure. This decision would be irrelevant if agents were immortal, there would be no population

[2]A CES specification is chosen for the simulations. "Model 3" does not have a labour/leisure choice but this aspect is addressed in other models of the paper.

growth and taxes would be lump-sum. Nevertheless with these features present, the financing decision implies changes in the long run rate of growth.

In Krichel and Levine (1996) the authors refine that work. On the modeling side, they introduce adjustment costs arising from changes in the private and public capital stock. The main motivation is here that it allows for an additional time-inconsistency effect. It turns out that any realistic calibration of the finite life and population growth aspect has hardly any impact on the steady-state interest rate and growth rates. The assumption of adjustment costs makes the model altogether more inelastic, i.e. policy has a smaller impact on the economy. In the simulation exercises, all the government is allowed to do is to vary taxes and the fraction of government expenditure spent on infrastructure.

As far as the difference between time consistent and optimal policy is concerned, both Krichel and Levine (1995) and Krichel and Levine (1996) reach the same conclusion. Although the time-consistent equilibrium is sub-optimal in terms of steady-state welfare, it yields *higher* growth, through an accumulation of assets by the state and a cut of government consumption.

In an interesting paper, Benhabib and Velasco (1996) examine the issue of optimal and time-consistent taxation. They study an infinitely-lived consumer in a small open economy with perfect capital mobility. This simplifies the analysis a great deal by removing the dynamics of the post-tax interest rate, since taxation is source based. They generalise the Barro (1990) production function by using a CES rather than Cobb-Douglas. In the first period is is optimal to set the tax rate to some optimising value say $\mathring{\tau}(0)$ for a given capital stock. In the next period, the government sets the tax rate to another value say $\mathring{\tau}$, that takes into account that the imposition of tax will distort the supply of capital. In the Barro model, $\mathring{\tau}(0) = \mathring{\tau}$ such that the economy would always be in a balanced growth path. But with the CES production function, the two solutions are not the same. The time-inconsistent solution is $\mathring{\tau}(0)$ for the first period, and some other $\mathring{\tau}$ for all others. In the time-consistent solution, the first tax rate $\mathring{\tau}(0)$ will always prevail in any period, because of the reoptimisation imposed to treat the current capital as given. The time-consistent path will maximise national output and hence growth in every period, but the time inconsistent path will lead to higher welfare, not because there is government consumption spending like in many other contributions in this strand of the literature but because the production function is not of a substitution elasticity of 1. In fact if the elasticity is larger that 1, we have $\mathring{\tau}(0) > \mathring{\tau}$ and vice versa. In this model, the time-consistent policy is welfare reducing for all parameter choices. The authors investigate the best sustainable rule using the trigger strategy concept. All the results of this paper are dependent on output depending on the flow of public investment, rather than the stock of capital. Otherwise with constant returns to scale, the current production depends on decisions taken in the previous period, the problem becomes genuinely dynamic and, alas, difficult to solve.

There is a whole strand of the literature that models the impact of the government's action on the economy's growth rate in a more indirect way. These studies assume that there is a private externality through which the productivity of capital remains bounded away from zero. These studies usually use the concept of human capital to introduce this externality. The government's policy can has a more indirect influence on the economy. Recent papers in that strand include Liu (1994), Ni and Wang (1994), Tran-Nam, Truong, and Ninh Van Tu (1995), Wang and Yip (1995), Greiner (1996), Martin and Rogers (1997) and Ihori (1997) for the closed economy and Osang and Pereira (1996) for a small open economy. It would be too long (and too boring) to review this strand of the literature here.

2.3 Differential growth in open economies

In Subsection 4.2 and Section 6 thesis I am particularly interested in the occurance of differential growth. The main question I address (and actually solve) is the existence of equilibria with differential growth. In other words: is it possible for one economy to grow faster than another permanently? Growth differentials are neglected in the current open economy macroeconomics literature. There are three reasons for that.

First the traditional open economy macroeconomics model is based on the basic Mundell-Fleming framework. It is not suited to the introduction of growth differentials, because growth only plays a limited rôle in the model. To take a recent example, van Tuijl, de Groof, and Koolnaar (1997) have a model that only uses exogenous technical progress to investigate spillovers of public capital and fiscal policy. Their model can address the impact of one economy's fiscal policy on the other along an exogenously given long-run growth path. This is technically introduced as by considering the deviation of each variable from the steady-state growth path using a linear model throughout. This long run exogenous nature of growth is not limited to this type of linear models. It is generic to most models that do not have explicit microfoundations.

Second, within the more microfoundation-based Swan (1956)-Solow (1956) framework of exogenous growth, the long run growth rate of GDP per capita is zero as long as there is no exogenous rate of technical progress. With a common technology, the question of diverging growth rates does not appear. If all countries have the same production function, and if preferences are identical[3] then countries will converge to the same level of GDP per capita. A recent paper that continues this exogenous growth approach is Barro, Mankiw, and Sala-i-Martin (1995). Production is a Cobb-Douglas with human capital, physical capital and raw labour as inputs. A small economy jumps to the steady state once opened to a world where interest rates are constant and capital mobility is perfect. To remove that feature, the authors introduce imperfect capital mobility in the sense that only physical capital can serve as a collateral for international borrowing and concentrate on the case where this constraint is binding when the economy is opened up to the international capital market. In that case only physical capital jumps to its steady state, human capital still takes time to adjust. This credit-constraint open economy then converges to a steady state in much the same way as a closed economy would do.

[3] There is no obvious reason why at the level of aggregation required for macroeconomic analysis tastes should not be the same.

Clearly the emphasis of macroeconomic theory on models in which the long run rate of growth is equal across countries contrast with the received wisdom of large—maybe growing—income disparities in the world[4]. With the advent of the "endogenous growth literature", the potential for diverging growth rates within models using optimizing agents has appeared. An important obstacle is the assumption of capital mobility. With perfect capital mobility, the equalisation of interest rate will imply that the marginal productivity of capital is is the same in all countries even in the short run. In many models, including endogenous growth models, this implies the equality of income in the long run, where factors are mobile. This is well illustrated on a textbook level by Barro and Sala-i-Martin (1995).

One of the first two-country endogenous growth models was Alogoskoufis and van der Ploeg (1991). They build a two-country model with perfect capital mobility and cross-country externalities of the capital stocks. Both countries produce an identical commodity. Capital provides for an externality on labour productivity, both domestically and internationally such that output can be written as a function of the domestic and foreign capital stock only. Since both are reproducible factors long-run growth will be possible. With perfect capital mobility, there will be equalization of growth rates in each period. This follows from the assumption that the technology is identical and from the assumption that all production factors can be traded. Note however that the authors allow for a difference in the scale parameter of the production function, therefore the model does not have level convergence. Growth rates remain common from an initial level of income that may be different, and incomes involve in parallel.

One important strand of the endogenous growth literature that has been used for multi-country work was pioneered by Grossman and Helpman (1992) which in turn is based on Romer (1989). In these models the growth process is modelled as the expansion of product varieties. There is no physical capital as such, production is a function of labour and knowledge. Knowledge capital is created as a by-product of R&D activity. By a convenient choice of units, the total capital stock can be considered as the sum of varieties produced in each country. Recent examples of this approach are Rivera-Batiz and Romer (1991), Devereux and Lapham (1994), Currie, Levine, Pearlman, and Chui (1998) and Wälde (1996). All rely on an identical utility function in both countries in the form of a discounted logarithmic felicity in a composite commodity. With this specific functional form, the interest rate in each instant is pinned down by the discount rate. Wälde (1996) studies the case of initially differing capital stocks, i.e. one country producing more varieties when both economies are closed. When the economies open, both will converge to a common growth rate since the stock of knowledge spreads to both countries. The rate of innovation in the long run depends on the total stock of knowledge capital. However in the country where innovation was slow, the rate initially overshoots, i.e. rises beyond the common long run rate and approaches it from above. Note that the common long run rate of growth is the result of international spillover of

[4] I will not attempt to summarize the large a empirical literature on the question of convergence.

knowledge capital. Devereux and Lapham (1994) had shown before that if there was no international spillover of knowledge, then the opening of economies will result in two countries with diverging size, as soon as the the initial knowledge stocks are not identical. In the limit one economy will do all the R&D and the other will do none, its production will be of size zero in the world economy, despite perfect capital mobility. This model is an early example for a model that has diverging growth rates.

Another strand in the literature that does allow for persistent growth differentials is Buiter and Kletzer (1991), Buiter and Kletzer (1993) and Buiter and Kletzer (1995). They have models that generate persistent growth differentials despite perfect capital mobility. The basic idea is that there is a source of growth that must be home-grown, i.e. cannot be imported. In their examples this home-grown input is human capital. All three papers are based on a three-period overlapping generations model that allows for proper modeling both of the process of accumulation of human capital within a generation as well as the transmission of human capital between generations. The three papers develop variations of the human capital accumulation process and the authors take great care to model this in precise details. One important message of these papers however is that if production involves a non-traded input, and if endogenous growth occurs, then there can be differences in labour productivity. Note that difference in labour productivity also implies steady-state divergence in the growth rate. This on the other hand implies that in the steady state, one country has an infinite size vs. the other. None of the papers formally addresses that issue. All calculations examine the short-run effects of policies that depart from an initial symmetric equilibrium. Thus it is not clear what impact differential growth rates have on the accumulation of assets in the longer run.

Is a growth-run growth differential possible? To simplify, identify "long run" with perpetual, and think of two-country framework. An intuitive first argument is that such a situation is not possible. If two economies grow at different rates, perpetually, then the slower growing economy will have a size zero in the long run, therefore we are implicitly studying a closed economy. This simple argument is probably the main reason why the academic literature has not addressed the possibility of long run growth differentials. I will refer to this idea as the "size argument".

One possible avenue for overcoming the above argument is to refine the notion of "size". If the size of an economy is total GDP, then there is a possibility that total GDP growth is uneven but compensated by a differential in the population growth rate. Razin and Yuen (1996a) and Razin and Yuen (1996b) use endogenous population growth to overcome this version of the size problem. However the problem with endogenous fertility is that over time the "world" population will only be in one country, the other will have relative population of zero and studying its characteristics will not be relevant.[5]

To summarise this literature, I conjecture that growth rate differentials do not appear in models where there is a direct spillover effect in factor productivity, for

[5]More on the economics of divergent population growth can be found in an interesting paper by Deardorff (1994).

example where there is diffusion of knowledge or where the capital stock in one country has an impact on the productivity of factors abroad. My conjecture would be that is is possible to have divergent growth as soon as there is one single home-grown production factor that can not be imported and the productivity of which is independent of factors abroad. In all existing models of differential growth, the long run is characterized by the slow growing country being of size zero.

In Subsection 4.2 I argue that long run growth differential can be sustained without any problems of either physical economy size or foreign asset accumulation. The key to this result is to consider a world where each country specializes in the production of its own good. In that case, the difference in GDP growth can be "compensated" by a change in the terms of trade. I show conditions under which a balanced growth path with differential growth exists and derive some of its properties.

2.4 The literature on debt and the de Silhouette problem

In this section, I am interested in the issue of financing a given stream of expenditure through either taxation or debt. There are situations in which this problem does not matter. This requires that the public sector is infinitely lived or that it is composed by overlapping generations linked by bequests. It also requires that taxes are not distortionary, i.e. typically lump-sum. Here I will be looking at any model where one of these conditions does not hold and this so-called "Ricardian equivalence" fails. In this case the choice between tax and debt financing does matter.

Probably the most famous contribution on the effects of public debt is Diamond (1965). He combines a neo-classical production function with an overlapping generations model to examine one of the reasons why Ricardian equivalence breaks down, the fact that consumers have finite lives and are not linked through bequests. To simplify matters, taxes are lump-sum. Diamond (1965) considers the situation where the debt per head is raise in period one, and then held constant at that higher level. The first generation will benefit from an increase in debt because with unchanged government spending the rise in debt means a decrease in taxation. For the next generation the increase in taxes caused by the increase in debt makes for a fall in welfare, but the interest-rate rise that one can expect (under some regularity condition) will introduce a price effect that may increase utility, thus the overall effect is ambiguous. However in the steady state, welfare will be lower for all generations, as long as the interest rate is larger than the growth rate.

What Diamond did not consider is the normative problem of a government maximising welfare. From the discussion of the impact of an exogenous increase in debt, it is clear that if the government does not discard the future very heavily it will decrease debt. In fact it can be shown that at around the point where debt is zero, a further reduction still improves welfare in the longer run, as long as the economy is dynamically efficient[6]. We can therefore conclude that fiscal policy should be viewed as an instrument to bring the economy towards the golden rule. If the economy is

[6] See Krichel (1997) for further details.

initially efficient, bringing the economy closer to the golden rule will imply that the government raises a stock of assets. It may not do this to the full extent as to bring the interest rate to the level of the growth rate because it discounts future benefits vs. current costs but in general we should expect it to bring both rates closer together by holding negative debt. Note this conclusion based on Diamond (1965) is independent of the idea that taxation is directly costly, because in his model taxation is lump-sum. It is also worth pointing out that although his conclusions are drawn for an ad hoc overall fiscal policy that stabilises debt after the first period, the conclusion should be robust for a wide variety of fiscal policies.

Another strand of the literature can be described as the tax-smoothing literature, and started with Barro (1979). His departure from the Ricardian equivalence results from assuming that taxation is distortionary. He seeks the optimal fiscal policy and finds that the intertemporal marginal tax distortions must be equalised through all periods. Roughly speaking that means that that debt should rise when the economy is hit by an adverse shock and decline when there is a favourable shock to the economy. The underlying reason for that comes straight out of concavity assumptions over the private sector's lifetime utility. Overall in the long run, debt should be zero. Mankiw (1987) extended this literature to the collection of seigniorage. However the power of his model is limited because he assumes constant velocity of circulation. Another important paper in this strand—it extends the idea of debt smoothing to a stochastic world—is Lucas and Stokey (1983) that I have already discussed in Subsection 2.1 on page 8. Their title alludes to an important limitation of this literature, the fact that it does not include the presence of a taxable stock. If there is a stock that can be taxed, then the nature of the problem changes completely. At any period t, the stock at the beginning of the period was formed in period $t-1$, therefore taxing it in period t does not imply any distortion. Hence the optimal way to tax a stock over time is to levy a heavy tax on the stock in the beginning. At unchanged expenditure, the government will reduce debt to accumulate assets. This policy will avoid the levying of taxes in the future. This is welfare improving because taxes on stocks yet to be formed are distortionary. In a rational expectations equilibrium, they discourage the formation of the stock in the current period.

An early formal account of this idea is Chamley (1986). He used an infinitely lived household accumulating capital for a neoclassical production technology. He shows that under fairly general conditions, the optimal tax on capital is zero in the long run. Government expenditure is financed through assets accumulated in earlier periods.

Note that this profile of taxation is not limited to the taxation of capital but generic to the taxation of any stock. A telling example is Obstfeld (1991). He is interested in seigniorage collection, but rather than assuming that the velocity remains constant, he stipulates a general function where the stock of money demanded depends negatively on the expected rate of inflation. He searches for a time-consistent equilibrium and shows—under very general conditions—that this implies that the government in the long run finances the expenditure stream by a stock of assets that it has accumulated in the previous periods.

Let us take stock of those results. We have two departures from Ricardian equivalence. One is that lives are finite, the other is that taxes are distortionary. As long as the former problem occurs in an economy that is dynamically efficient, and as soon as the latter includes the option to tax a stock, both departures from Ricardian equivalence point to the same pattern of taxes and debt over time. This is an initial outburst of taxation leading to an accumulation of assets that finance the expenditure in future periods.

Note that this trajectory of fiscal policy does not directly rely on an assumption of the presence or absence of time consistency as it is sometimes alleged.[7] If the trajectory is time consistent, reoptimisation in every period is likely to lead to an increase in assets in every period as long as there are taxation requirements in the future[8]. In the case of a precommitment solution, this is not necessarily the case because the taxation cost in the future is discounted heavily vs. the cost of current tax increases, therefore in general under a precommitment regime, we would expect that the asset position worsens in the long run, though not necessary as much as to give positive debt.

It is useful to label the whole class of policies that are characterized by an initial sharp rise in taxation and accumulation of assets as a "de Silhouette" policy. I invented this term to honour Etienne de Silhouette[9], a former French finance minister who strongly believed in the optimality of this type of policy. His term of office as "contrôleur des finances" in 1756 was very brief indeed. He lost favour with the aristocracy when he planned to raise a tax on land.

Turnovsky (1996) provides an alternative interpretation of de Silhouette policies. His focus is a production function similar to Barro (1990), in particular the model is in the balanced growth path at any point in time. He also introduces congestion effects in both public consumption and investment. His focus is the comparison between the allocation achieved by a central planner and the decentralised equilibrium. He limits attention to either government consumption or government investment, i.e. each appears separately. Debt is only considered in the case where the government expenditure has no productive rôle. If a consumption tax can be levied, then the optimum can be implemented in the decentralised economy. If there is no consumption tax, then the optimum can only be implemented if the government is initially a creditor to the economy. Thus we can understand a de Silhouette policy as a means to direct a decentralised economy onto the optimum path.

Sadly, the de Silhouette theory of fiscal policy contrasts sharply with the what happens in the "Real World". Therefore the profession does tend to "lowlight" these results. For example, in a recent broad survey of fiscal policy Tanzi and Zee (1997) do not mention this issue at all. I think that it is important to think of the implications for practical policy of the welfare loss incurred through public debt.

[7]For an example see van der Ploeg (1995), page 439.

[8]This is in fact the case considered in Obstfeld (1991).

[9]Conventionally we think of a silhouette as an outline shade. In fact these drawings are named after him. They became fashionable in his time to ridicule the man and his policies.

	cooperation	non-cooperation
not time-consistent	SP $\approx$ TP	
time-consistent	ST $\approx$ TC	TN

Table 2.1: Overview of the regimes

Further work on simple calibrated models could give a rough estimate for the welfare loss.

2.5 The model of Devereux and Mansoorian

From a technical point of view, the work of this thesis is based on Devereux and Mansoorian (1992). This is a straightforward generalisation of Barro (1990) to a two-country world where each country is fully specialised in the production of a single traded commodity. National production is a function of the domestic capital stock and the domestic infrastructure stock. This makes for a non-traded factor without spillover effect, therefore uneven growth is possible. Households wish to consume commodities from both countries at fixed budget shares. There is complete depreciation of both capital stocks. Symmetry between policymakers implies the same policies are pursued in each country. Policy choices are static. Therefore the economies are always in a balanced steady state. In a Nash equilibrium countries choose their taxes and spending rates independently from each other to maximise domestic welfare. In the cooperative optimum a central planner maximises the sum of the welfare of domestic and foreign welfare. The first result of the paper, their proposition 1, is that the choice of government investment is not subject to a coordination problem. In other words, the spending on government infrastructure is identical under Nash and cooperative regimes, and identical to the one in Barro (1990), discussed on page 70 below. When government consumption is taken account of the results depends crucially in the elasticity of substitution in the felicity function. With logarithmic felicity, an analytical solution is possible and it can be shown that cooperative equilibrium involves a decline in public consumption and an increase in growth. This result carries through for a "high" (i.e. larger than 1) elasticity of substitution. But if the elasticity is low, then this result may reverse. In that case cooperation involves slower growth and higher taxes than the Nash equilibrium. The received wisdom is that the elasticity is about .5, in which case the ambiguity is very important. When the elasticity is .4 the two regimes are virtually identical.

The emphasis of Devereux and Mansoorian (1992) is on optimal policies in a model that is simple enough to allow for such policies to be computed. For simplicity they look at a totally symmetric world. In that case both countries will follow the same policy. In this thesis my first interest are asymmetric policies. What happens if the two governments set taxes rates that are not the same? Then surely one economy will grow at a different rate than the other. Is there an equilibrium? Will such an equilibrium be reached if one government decides to change is taxes, departing from an initially symmetric equilibrium?

The model of Devereux and Mansoorian (1992) is always on a balanced growth path. This thesis sets out a model that is more general. Most importantly, I allow for incomplete depreciation of capital. This makes for a genuinely *dynamic* model. In such a setting the problem of time inconsistency arises and it is related to the issue of policy coordination. It is easiest to consider the two issues separately. A summary of the regimes is given in Table 2.5. To carefully isolate the issue of optimal vs time-consistent policy I explicitly construct a single-country model and use simulations for that model to investigate that issue in Section 7. There I distinguish a Single country Precommitment regime SP and a Single country Time-consistent regime ST. The single country simulations are the same than those obtained for the two-country cooperative regime, where they are named TP (Two-country Precommitment) and TC (Two-country Cooperative). The Non-cooperative time-consistent regime is called TN. Regimes TN, TC and TP are all discussed in Section 8, but the emphasis is on the comparison between TC and TN.

A generalisation of models of the type of Barro (1990) for one country and Devereux and Mansoorian (1992) for two countries is an obvious choice for the topics of the thesis because

- in these models the private sector and the public sector are both treated as infinitely lived agents, which makes the treatment logically more transparent;
- both the private and the public sector have the same problem to solve, essentially a repeated consumption/savings problem;
- the simultaneous presence of public consumption and an infrastructure stock as a crucial input in production makes for an important rôle of fiscal policy both in the short run and the long run. This relationship is exactly what we need to focus on for the study of time consistency;
- since in each country there is a capital input that must be home-grown i.e., that can not be imported, there is an opportunity to study growth differentials;
- the models are based on very standard formulations of utility and production and they are therefore not likely to lead to technical difficulties that would distract from the essential message conveyed by the model.

Therefore I am introducing such a model in the next section.

3 The Model

The model is a generalisation of the model of Devereux and Mansoorian (1992). I add three features. First the shares of consumption of the domestic and foreign commodity in utility do not need to be 1/2. Second and most importantly, I allow for incomplete depreciation of stocks at rate δ. This feature makes the model *dynamic*. Last, I model infrastructure as a stock rather than a flow. In Subsection 3.1 I deal with the demand side of the model. The microfoundations of the demand side are very important for the model. The forward-looking nature of consumption causes time-inconsistency of policy that will be studied in more detail later. To allow for a more transparent treatment of this issue, time is discrete, and points in time are denoted $\mathfrak{t}$. Period $\mathfrak{t}$ stretches between date $\mathfrak{t}-1$ and $\mathfrak{t}$. This implies that stocks are noted as end-of-period magnitudes.

3.1 Consumption of infinitely lived household

Let there be two economies called Home and Foreign. Both are specialised in the production of a traded commodity called the "home commodity" and the "foreign commodity", respectively. Both commodities are used for consumption both at home and abroad; the domestic commodity is also used for capital accumulation in its respective country. Let C_{d} be the consumption of the home commodity at Home, and C_{f} the consumption of the foreign commodity at Home. Let there be a single infinitely-lived representative consumer in both the domestic and the foreign economy. Both have the same type of utility function. For the domestic economy we have

$$U(\mathfrak{t}) = \sum_{\mathfrak{t}'=0}^{\infty} \varrho^{\mathfrak{t}'}\, u(C_{\mathrm{d}}(\mathfrak{t}+\mathfrak{t}'), C_{\mathrm{f}}(\mathfrak{t}+\mathfrak{t}'), G^{\mathrm{c}}(\mathfrak{t}+\mathfrak{t}')) \tag{3.1}$$

where felicity takes the isoelastic form

$$u(\cdot) = \frac{\left(C_{\mathrm{d}}(\mathfrak{t})^{\alpha}\, C_{\mathrm{f}}(\mathfrak{t})^{1-\alpha}\right)^{1-\sigma} - 1}{1-\sigma} + \eta\, \frac{G^{\mathrm{c}}(\mathfrak{t})^{1-\sigma} - 1}{1-\sigma} \tag{3.2}$$

As is standard is this type of literature, (3.1) implicitly assumes additive separability of the utility in different periods. Utility U is a discounted sum of the felicity u derived from consumption in different periods. $0 < \varrho < 1$ is the discount factor. When convenient for the clarity of exposition, I also use the discount rate д defined by $\varrho = 1/(1+\text{д})$.

The isoelastic nature of the felicity function in (3.2) is also a standard feature; the elasticity of substitution is given by $1/\sigma$. η is a parameter that indicates the importance of publicly provided consumption commodity G^{c}. As noted by Devereux and Mansoorian the assumption that utility from consumption in different periods can be additively separated is important for the model. It ensures analytical simplicity and prevents the government from manipulating consumers' intertemporal elasticities through changes in fiscal policy. I follow Barro (1990) and Devereux and Mansoorian (1992) by assuming that the publicly provided commodity is not subject to a congestion effect. According to authors who have explicitly introduced congestion effect—Glomm and Ravikumar (1994) or Turnovsky (1996)—this implies that usage of the commodity by one consumer does not impede on the usage by another person, i.e., that the commodity would be non-rival. However compelling this argument may be, it fails to convince in this model where the private sector is already modelled as one unit. Its size remains at unity in every period.

The budget identity of a consumer at date t is:

$$\begin{aligned} P(\mathsf{t})\, C_{\mathrm{d}}(\mathsf{t}) + P^{*}(\mathsf{t})\, C_{\mathrm{f}}(\mathsf{t}) + P(\mathsf{t})\, Л(\mathsf{t}) + P(\mathsf{t})\, K(\mathsf{t}) \\ = P(\mathsf{t})\, Y(\mathsf{t}) + P(\mathsf{t})\, Л(\mathsf{t}-1)\,(1 + r(\mathsf{t}-1)) + (1-\delta)\, P(\mathsf{t}-1)\, K(\mathsf{t}-1) \end{aligned}$$

Here $Л$ is international lending by the Home consumer, expressed in units of the Home commodity and K is Home capital in units of the domestic commodity. P and P^* are the prices of Home and Foreign commodity respectively. The last term would disappear in Devereux and Mansoorian's model since they implicitly assume $\delta = 1$. I use the relative price as

$$p(\mathsf{t}) = \frac{P^{*}(\mathsf{t})}{P(\mathsf{t})}$$

Home non-human wealth $Ш^{a}(\mathsf{t})$ is expressed in terms of the Home commodity as

$$Ш^{a}(\mathsf{t}) = K(\mathsf{t}) + Л(\mathsf{t}) \tag{3.3}$$

and Home consumption expenditure is defined as

$$C(\mathsf{t}) = C_{\mathrm{d}}(\mathsf{t}) + p(\mathsf{t})\, C_{\mathrm{f}}(\mathsf{t}) \tag{3.4}$$

in terms of the Home commodity. Using these conventions, we can write the budget identity for the domestic economy as

$$C(\mathsf{t}) + Л(\mathsf{t}) + K(\mathsf{t}) = Y(\mathsf{t}) + (1 + r(\mathsf{t}-1))\, Л(\mathsf{t}-1) + (1-\delta)\, K(\mathsf{t}-1) \tag{3.5}$$

In this model, *firms*, not households are taxed. Households receive a disposable income Y from firms that has been taxed:

$$Y(\mathsf{t}) = Q(\mathsf{t})\,(1 - \tau(\mathsf{t}))$$

Here Q is domestic output. Income could be split into income from labour and capital, both being taxed at the same level as in

$$Y(\mathsf{t}) - \delta\, K(\mathsf{t}-1) = w(\mathsf{t})\, L(\mathsf{t}) + r(\mathsf{t}-1)\, K(\mathsf{t}-1) \tag{3.6}$$

where L is labour (considered a flow here) and Q is output. In factor market equilibrium w (the post-tax wage) will be equal to the post-tax marginal product of labour (including externalities) and $\delta + r$, where r is the post-tax interest rate, will be equal to the private marginal productivity of capital as expected in period $\mathsf{t}-1$. Using (3.6) and (3.3) in (3.5) gives the budget identity

$$C(\mathsf{t}) + Ш^{a}(\mathsf{t}) = w(\mathsf{t})\, L(\mathsf{t}) + (1 + r(\mathsf{t}-1))\, Ш^{a}(\mathsf{t}-1) \tag{3.7}$$

I now need to define the interest rate between period $\mathfrak{t}$ and $\mathfrak{t}'$.

$$r_{\mathfrak{t}}(\mathfrak{t}') = \max\left(\prod_{\mathfrak{t}''=\mathfrak{t}}^{\mathfrak{t}'}\left(1 + r(\mathfrak{t}''), 1\right)\right) - 1$$

If the usual limiting condition holds

$$\lim_{\mathfrak{t}'\to\infty} \frac{Ш^{a}(\mathfrak{t}'+1)}{1 + r_{\mathfrak{t}-1}(\mathfrak{t}')} = 0$$

we can forward (3.7) into the future to get

$$Ш(\mathfrak{t}-1) = \sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{C(\mathfrak{t}')}{1 + r_{\mathfrak{t}-1}(\mathfrak{t}'-1)} \tag{3.8}$$

Here $Ш$ is total wealth, the sum of non-human (or asset) wealth and human wealth

$$Ш(\mathfrak{t}-1) = K(\mathfrak{t}-1) + Л(\mathfrak{t}-1) + Ш^{h}(\mathfrak{t}-1) \tag{3.9}$$

where $Ш^{h}(\mathfrak{t}-1)$ is human wealth at the beginning of $\mathfrak{t}$, i.e. the present value of current and future wage earnings

$$Ш^{h}(\mathfrak{t}-1) = \sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{w(\mathfrak{t}')\, L(\mathfrak{t}')}{1 + r_{\mathfrak{t}-1}(\mathfrak{t}'-1)} \tag{3.10}$$

The consumer's problem is to maximise (3.1) under the constraint (3.8), with which we associate the multiplier $\lambda(\mathfrak{t})$. The first order conditions[10] imply

$$C(\mathfrak{t}') = \frac{1}{\alpha}\, C_{\mathrm{d}}(\mathfrak{t}') = \frac{1}{1-\alpha}\, C_{\mathrm{f}}(\mathfrak{t}') \qquad \forall\, \mathfrak{t}' \geq \mathfrak{t} \tag{3.17}$$

and

$$C(\mathfrak{t}') = \left(\frac{p(\mathfrak{t})}{p(\mathfrak{t}')}\right)^{\frac{(1-\sigma)\,(1-\alpha)}{\sigma}} C(\mathfrak{t})\, \frac{(1+\text{д})^{(\mathfrak{t}-\mathfrak{t}')/\sigma}}{(1+r_{\mathfrak{t}}(\mathfrak{t}'-1))^{-1/\sigma}} \tag{3.18}$$

Equation (3.17) implies that the budget shares of both commodities are constant, a well known property of Cobb-Douglas felicity. Equation (3.18) gives the evolution of total consumption as a function of initial consumption. If $\mathfrak{t}' = \mathfrak{t}$, (3.18) becomes an identity. If $\sigma = 1$, the first term becomes a constant and any relative price price change will not impact on the time profile of consumption. If the degree of intertemporal substitution is high—$\sigma < 1$, then as the terms of trade improve, the consumption profile tilts downwards and current consumption rises. In the conventional case where the degree of intertemporal substitution is low i.e., $\sigma > 1$, then as the terms of trade improve, the consumption profile tilts upwards and current consumption falls. Even the terms of trade remain constant, the elasticity of substitution is still important because it determines how sensitive consumption is to changes in the interest rate and the discount rate. To pin down the current level

[10] For any period $\mathfrak{t}'$ we have

$$\varrho^{\mathfrak{t}'} \frac{\partial u}{\partial C_{\mathrm{d}}(\mathfrak{t}')} = \frac{\lambda(\mathfrak{t})}{1+r_{\mathfrak{t}-1}(\mathfrak{t}'-1)} \tag{3.11}$$

and in the next period

$$\varrho^{\mathfrak{t}'+1} \frac{\partial u}{\partial C_{\mathrm{d}}(\mathfrak{t}'+1)} = \frac{\lambda(\mathfrak{t})}{1+r_{\mathfrak{t}-1}(\mathfrak{t}')} \tag{3.12}$$

These equations also hold for the foreign good

$$\varrho^{\mathfrak{t}'} \frac{\partial u}{\partial C_{\mathrm{f}}(\mathfrak{t}')} = \frac{\lambda(\mathfrak{t})\, p(\mathfrak{t}')}{1+r_{\mathfrak{t}-1}(\mathfrak{t}'-1)} \tag{3.13}$$

Using the specifics of the felicity function

$$\begin{aligned} \frac{\partial u}{\partial C_{\mathrm{d}}} &= \alpha\, C_{\mathrm{d}}^{\alpha\,(1-\sigma)-1}\, C_{\mathrm{f}}^{(1-\alpha)\,(1-\sigma)} \\ \frac{\partial u}{\partial C_{\mathrm{f}}} &= (1-\alpha)\, C_{\mathrm{d}}^{\alpha\,(1-\sigma)}\, C_{\mathrm{f}}^{(1-\alpha)\,(1-\sigma)-1} \end{aligned} \tag{3.14}$$

Using (3.14) in (3.11) and (3.12) we get

$$\varrho \left(\frac{C_{\mathrm{d}}(\mathfrak{t}'+1)}{C_{\mathrm{d}}(\mathfrak{t}')}\right)^{\alpha\,(1-\sigma)-1} \left(\frac{C_{\mathrm{f}}(\mathfrak{t}'+1)}{C_{\mathrm{f}}(\mathfrak{t}')}\right)^{(1-\alpha)\,(1-\sigma)} = \frac{1}{1+r(\mathfrak{t}')} \tag{3.15}$$

Using (3.14) in (3.11) and (3.13) and dividing, we find

$$p(\mathfrak{t}')\, \alpha\, C_{\mathrm{f}}(\mathfrak{t}') = (1-\alpha)\, C_{\mathrm{d}}(\mathfrak{t}') \tag{3.16}$$

using (3.4) in (3.16) and using (3.16) and (3.4) in (3.15) we get (3.17) and (3.18) of consumption, I need to substitute the sequence (3.18) in the budget constraint (3.8). This leads to

$$C(t) = \frac{Ш(t-1)\,(1+r(t-1))}{1+\xi(t)} \tag{3.19}$$

where I have introduced the expression

$$1+\xi(t) = \sum_{t'=t}^{\infty} \left(\frac{p(t)}{p(t')}\right)^{\frac{(1-\sigma)\,(1-\alpha)}{\sigma}} \frac{(1+д)^{(t-t')/\sigma}}{(1+r_t(t'-1))^{1-1/\sigma}} \tag{3.20}$$

to shorten notation. Changes in this term are crucial for the dynamics of the model, and in particular to the distinction of the closed and open economy. I call ξ the "reluctance rate". The larger the reluctance rate, the smaller is the rise in consumption that follows an increase in wealth. The evolution of $\xi(t)$ is given by

$$\xi(t) = \left(\frac{p(t)}{p(t+1)}\right)^{\frac{(1-\sigma)\,(1-\alpha)}{\sigma}} \frac{1+\xi(t+1)}{(1+r(t))^{1-1/\sigma}\,(1+д)^{1/\sigma}} \tag{3.21}$$

If $\sigma = 1$ there is no impact of relative prices on the reluctance to consume. When $\sigma > 1$ then if the domestic terms of trade improve, there will be an increase in the reluctance to consume. In the less conventional case where $\sigma < 1$ an improvement in the term of trade decreases the reluctance rate and results in higher consumption.

This completes the demand side of the model.

3.2 Other aspects of the model

I depart from a standard neoclassical production function where output is written as a constant returns to scale production function in accumulated capital and labour. For simplicity, I adopt the Cobb-Douglas formulation

$$Q(t) = \bar{\epsilon}\,K(t-1)^{\kappa}\,(\epsilon(t)\,L(t))^{1-\kappa} \tag{3.22}$$

where κ is capital's share in output, and ϵ is the efficiency of raw labour. Following Krichel and Levine (1996), I assume in turn that the efficiency of labour depends on the ratio of an aggregate index of capital per labour

$$\epsilon(t) = \frac{K(t-1)^{\gamma'}\,K^{g}(t-1)^{1-\gamma'}}{L(t)} \tag{3.23}$$

where K^{g} is a stock of infrastructure provided by the government. Substitute (3.23) into (3.22) to get

$$Q(t) = \bar{\epsilon}\,K(t-1)^{\gamma}\,K^{g}(t-1)^{1-\gamma} \tag{3.24}$$

for an appropriate γ. Since the interest rate is expressed in the respective countries' commodities, and depreciation is not tax deductible, the profit maximising condition implies

$$\delta + r(t) = \bar{\epsilon}\,(1-\tau(t+1))\,\kappa\,K(t)^{\kappa-1}K^{g}(t)^{1-\kappa} \tag{3.25}$$

Labour income includes the product of the externality[11]

$$w(t)\,L(t) = (1-\tau(t))\,Q(t) - (\delta + r(t-1))\,K(t-1)$$

From (3.25) lagged, we get

$$w(t)\,L(t) = (1-\tau(t))\,(1-\kappa)\,Q(t) \tag{3.26}$$

Therefore human wealth accumulates as

$$Ш^{h}(t) = (1 + r(t-1))\,Ш^{h}(t-1) + (1-\tau(t))\,(1-\kappa)\,Q(t) \tag{3.27}$$

Each country's capital is composed out of domestic output only. It evolves according to

$$K(t) = (1-\delta)\,K(t-1) + I(t)$$

In the same way the government capital stock is only made of domestic output. It evolves according to

$$K^{g}(t) = (1-\delta)\,K^{g}(t-1) + G^{i}(t)$$

The commodity market clearing conditions are

$$\begin{aligned} Q(t) &= G(t) + I(t) + C_{d}(t) + C_{d}^{*}(t) \\ Q^{*}(t) &= G^{*}(t) + I^{*}(t) + C_{f}(t) + C_{f}^{*}(t) \end{aligned} \tag{3.28}$$

The consumption demands for each commodity can be linked to the consumption expenditures. Express each consumption level in the units of the commodity it concerns, not in units of the commodity of the country it originates from. Applying (3.17) I get

$$\begin{aligned} C_{d}(t) &= \alpha\,C(t) \\ C_{f}(t) &= \frac{1-\alpha}{p(t)}\,C(t) \\ C_{f}^{*}(t) &= \alpha^{*}C^{*}(t) \\ C_{d}^{*}(t) &= (1-\alpha^{*})\,p(t)\,C^{*}(t) \end{aligned} \tag{3.29}$$

where α^{*} is the share of the foreign commodity in foreign welfare function. For government spending, I initially impose that all spending is tax financed

$$G(t) = \tau(t)\,Q(t) \tag{3.30}$$

[11]It is possible to be more rigorous and introduce the proceeds from the externality as equity held by the private sector, see for example Benhabib and Velasco (1996). However that will make no difference to the results that I present as long as the income from the shares is taxed at the same rate as other sources of income

Relaxing the (3.30) implies complications that require special attention. This issue is addressed in Section 9. For now substitute (3.30) and (3.29) in the commodity market clearing conditions

$$(1-\tau(\mathsf{t}))\,Q(\mathsf{t}) = I(\mathsf{t}) + \alpha\, C(\mathsf{t}) + (1-\alpha^*)\, p(\mathsf{t})\, C^*(\mathsf{t}) \tag{3.31}$$

$$(1-\tau^*(\mathsf{t}))\,Q^*(\mathsf{t}) = I^*(\mathsf{t}) + \alpha^* C^*(\mathsf{t}) + \frac{1-\alpha}{p(\mathsf{t})}\, C(\mathsf{t}) \tag{3.32}$$

To find the terms of trade, generate the world goods market equilibrium by adding (3.31) to (3.32) to get

$$p(\mathsf{t}) = -\frac{(1-\tau(\mathsf{t}))\,Q(\mathsf{t}) - I(\mathsf{t}) - C(\mathsf{t})}{(1-\tau^*(\mathsf{t}))\,Q^*(\mathsf{t}) - I^*(\mathsf{t}) - C^*(\mathsf{t})} \tag{3.33}$$

We recognise in the denominator the trade balance of the domestic economy and in the numerator the trade balance of the foreign economy. If this is zero then $p(\mathsf{t})$ could be any real number. Thus we need to explicitly model an asymmetry in the trade balance if we wish to maintain a defined price. Foreign lending accumulates with the current balance

$$\Pi(\mathsf{t}) - (1 + r(\mathsf{t}-1))\,\Pi(\mathsf{t}-1) = (1-\tau(\mathsf{t}))\,Q(\mathsf{t}) - I(\mathsf{t}) - C(\mathsf{t}) \tag{3.34}$$

$$\Pi^*(\mathsf{t}) - (1 + r^*(\mathsf{t}-1))\,\Pi^*(\mathsf{t}-1) = (1-\tau^*(\mathsf{t}))\,Q^*(\mathsf{t}) - I^*(\mathsf{t}) - C^*(\mathsf{t}) \tag{3.35}$$

Using (3.34) and (3.35) in (3.33) we get

$$-\frac{\Pi(\mathsf{t}) - (1 + r(\mathsf{t}-1))\,\Pi(\mathsf{t}-1)}{p(\mathsf{t})} = \Pi^*(\mathsf{t}) - (1 + r^*(\mathsf{t}-1))\,\Pi^*(\mathsf{t}-1) \tag{3.36}$$

and since the foreign lending of one country must be the foreign lending of the other or

$$\Pi(\mathsf{t}) = -p(\mathsf{t})\,\Pi^*(\mathsf{t}) \tag{3.37}$$

in each period, we have the condition that

$$\frac{p(\mathsf{t})}{p(\mathsf{t}-1)} = \frac{1 + r(\mathsf{t}-1)}{1 + r^*(\mathsf{t}-1)} \tag{3.38}$$

which, forwarded by one period is the familiar uncovered interest parity condition.

This completes the exposition of the model. The emphasis here has been on any individual country. But note that this is a $2 \times 2 \times 2$ model, with two countries two agents in each country, and two commodities.

Within each country, we have a model where economic policy is crucial for welfare. Through the accumulation of infrastructure, there is a direct impact of government spending on economic growth, but there is also an indirect impact of taxation on growth through a reduction in the post-tax return on investment.

Table 3.1: The complete model

To conserve space I start with equations for the domestic country that have obvious foreign counterparts.

$$C(\mathsf{t}) = \frac{\text{Ш}(\mathsf{t}-1)\,(1+r(\mathsf{t}-1))}{1+\xi(\mathsf{t})} \tag{3.39}$$

$$\text{Ш}(\mathsf{t}-1) = K(\mathsf{t}-1) + \text{Л}(\mathsf{t}-1) + \text{ш}^{\mathrm{h}}(\mathsf{t}-1) \tag{3.40}$$

$$K(\mathsf{t}) = (1-\delta)\,K(\mathsf{t}-1) + I(\mathsf{t}) \tag{3.41}$$

$$\text{Л}(\mathsf{t}) = (1+r(\mathsf{t}-1))\,\text{Л}(\mathsf{t}-1) + (1-\tau(\mathsf{t}))\,Q(\mathsf{t}) - I(\mathsf{t}) - C(\mathsf{t}) \tag{3.42}$$

$$\text{ш}^{\mathrm{h}}(\mathsf{t}) = (1+r(\mathsf{t}-1))\,\text{ш}^{\mathrm{h}}(\mathsf{t}-1) + (1-\tau(\mathsf{t}))\,(1-\gamma_2)\,Q(\mathsf{t}) \tag{3.43}$$

$$Q(\mathsf{t}) = \bar{\epsilon}\,K(\mathsf{t}-1)^{\gamma_2}\,K^{\mathrm{g}}(\mathsf{t}-1)^{1-\gamma_2} \tag{3.44}$$

$$K^{\mathrm{g}}(\mathsf{t}) = (1-\delta)\,K^{\mathrm{g}}(\mathsf{t}-1) + G^{\mathrm{i}}(\mathsf{t}) \tag{3.45}$$

$$G^{\mathrm{i}}(\mathsf{t}) = \text{г}(\mathsf{t})\,\tau(\mathsf{t})\,Q(\mathsf{t}) \tag{3.46}$$

$$G^{\mathrm{c}}(\mathsf{t}) = (1-\text{г}(\mathsf{t}))\,\tau(\mathsf{t}))\,Q(\mathsf{t}) \tag{3.47}$$

$$\delta + r(\mathsf{t}) = \bar{\epsilon}\,(1-\tau(\mathsf{t}+1))\,\gamma_2\,K(\mathsf{t})^{\gamma_2-1}K^{\mathrm{g}}(\mathsf{t})^{1-\gamma_2} \tag{3.48}$$

Next I list equations that are specific to the domestic country.

$$\xi(\mathsf{t}) = \left(\frac{p(\mathsf{t})}{p(\mathsf{t}')}\right)^{\frac{(1-\sigma)\,(1-\alpha)}{\sigma}} \frac{1+\xi(\mathsf{t}+1)}{(1+r(\mathsf{t}))^{1-1/\sigma}\,(1+\text{д})^{1/\sigma}} \tag{3.49}$$

$$I(\mathsf{t}) = (1-\tau(\mathsf{t}))\,Q(\mathsf{t}) - \alpha\,C(\mathsf{t}) - p(\mathsf{t})\,(1-\alpha^*)\,C^*(\mathsf{t}) \tag{3.50}$$

$$\text{Л}^*(\mathsf{t}-1) = -\text{Л}(\mathsf{t})/p(\mathsf{t}) \tag{3.37}$$

Finally this is a list of the equations that are specific to the foreign country

$$\xi^*(\mathsf{t}) = \left(\frac{p(\mathsf{t})}{p(\mathsf{t}')}\right)^{\frac{(1-\sigma^*)\,(1-\alpha^*)}{\sigma^*}} \frac{1+\xi^*(\mathsf{t}+1)}{(1+r^*(\mathsf{t}))^{1-1/\sigma^*}\,(1+\text{д}^*)^{1/\sigma^*}} \tag{3.51}$$

$$I^*(\mathsf{t}) = (1-\tau^*(\mathsf{t}))\,Q^*(\mathsf{t}) - \frac{(1-\alpha)\,C(\mathsf{t})}{p(\mathsf{t})} - \alpha^* C^*(\mathsf{t}) \tag{3.52}$$

4 The steady state of the model with and without growth divergence

In this section I investigate the steady state of the model. The Home economy grows at the rate $n > 0$, therefore the aggregates considered in Section 3 will not converge. Instead I consider all aggregates as ratios of GDP and introduce a lowercase notation like

$$\text{л}(\mathfrak{t}-1) = \frac{\text{Л}(\mathfrak{t}-1)}{Q(\mathfrak{t}-1)}$$

I first examine the steady state of the single country model in Subsection 4.1, before I turn to the steady state of the two-country model in Subsection 4.2.

$$c = \frac{\text{щ}\,(1+r)}{(1+n)\,(1+\xi)} \tag{4.1}$$

$$\text{щ} = \text{щ}^{\mathrm{h}} + k \tag{4.2}$$

$$\frac{1+\xi}{1+r} = \xi\,\left(\frac{1+\text{д}}{1+r}\right)^{1/\sigma} \tag{4.3}$$

$$k = \frac{1+n}{n+\delta}\,i \tag{4.4}$$

$$\text{щ}^{\mathrm{h}} = (1-\kappa)\,(1-\tau)\,\frac{1+n}{r-n} \tag{4.5}$$

$$1+n = \bar{\epsilon}\,k^{\gamma}\,k^{\mathrm{g}\,1-\gamma} \tag{4.6}$$

$$k^{\mathrm{g}} = \frac{g^{\mathrm{i}}\,(1+n)}{\delta+n} \tag{4.7}$$

$$g^{\mathrm{i}} = \tau\,\text{г} \tag{4.8}$$

$$g^{\mathrm{c}} = \tau\,(1-\text{г}) \tag{4.9}$$

$$\delta + r = (1-\tau)\,\kappa\,(1+n)/k \tag{4.10}$$

$$1-\tau = c+i \tag{4.11}$$

Table 4.1: The steady state of the single country

4.1 The steady state of the single country

To transform the model to a single economy, it suffices to set $p(\mathfrak{t}) = \alpha = 1$ and $\text{Л}(\mathfrak{t}) = \alpha^* = 0$. I consider a steady state where the economy grows at a fixed rate n in every period. It straightforward to show that such a steady-state is summarized by Table 4.1. It is less straightforward to see the following proposition.

Proposition 4.1 There is no steady state of the model with strictly positive consumption unless

$$1+r = (1+\text{д})\,(1+n)^{\sigma} \tag{4.12}$$

Proof 4.1 To how see that (4.12) must be satisfied, I proceed by contradiction and assume that n and r can take arbitrary values. Substitute $\mathfrak{w}^{\mathrm{h}}$ from (4.5) into (4.2), solve (4.10) for k and substitute in (4.2). Then substitute for $\mathfrak{w}$ in (4.1), to see that

$$c = \left[(1-\tau)\,\kappa\,\frac{1+n}{\delta+r} + (1-\kappa)\,(1-\tau)\,\frac{1+n}{r-n}\right]\frac{1+r}{(1+n)\,(1+\xi)} \tag{4.13}$$

A similar equation for investment can be found when solving (4.10) for k and substituting in (4.4)

$$i = \frac{(1-\tau)\,\kappa\,(n+\delta)}{r+\delta} \tag{4.14}$$

Now using the expressions for c from (4.13) and for i from (4.14) into (4.11), I find

$$1 = \frac{\kappa\,(n+\delta)}{r+\delta} + \left[\kappa\,\frac{1+n}{\delta+r} + (1-\kappa)\,\frac{1+n}{r-n}\right]\frac{1+r}{(1+n)\,(1+\xi)} \tag{4.15}$$

After some tedious algebra, it can be show that one solution to (4.15) is characterised by

$$\kappa = \frac{r+\delta}{n+\delta} \tag{4.16}$$

Substituting this equation back into (4.13) we obtain the result that $c = 0$. How this result comes about can be seen when we substitute (4.16) into (4.5) and (4.10). This leads to $k = -\mathfrak{w}^{\mathrm{h}}$, which means that $\mathfrak{w} = 0$. To see that (4.12) must be satisfied, first note that using (4.3), (4.12) is equivalent to

$$1+\xi = \frac{1+r}{r-n} \tag{4.17}$$

Then it suffices to substitute (4.17) in (4.15) to see that it becomes an identity. Q.E.D.

Equation (4.17) is readily interpreted as constraining long run marginal propensity to consume out of income to equal one in the long run. To see this, abstract from the distinction between human and physical wealth and consider that wealth is the discounted sum of private sector income, Y in every period. Then

$$\mathfrak{W}(\mathfrak{t}) = \sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{Y(\mathfrak{t}')}{1+r_{\mathfrak{t}}(\mathfrak{t}'-1)} \tag{4.18}$$

with steady growth and interest rates this becomes

$$\mathfrak{w} = y\,\frac{1+r}{r-n} \tag{4.19}$$

Using (4.19) and (4.17) in (4.1), I find $c = y$, which means that equation (4.12) constrains the long run propensity to consume out of income to 1.

4.2 An asymmetric steady state

In Section 2 on page 17, I conjectured that economies can grow at different rates all the time when there is an essential factor of production that is not traded. In the model of the thesis, infrastructure has that property. However, in all preceding models the size of the economy that is slow growing economy will vanish over time, i.e. it will become arbitrarily small in the world economy.

In the following, I make the assumption of differing growth rate and investigate that consequences of this assumption on the variables of the model. I allow the domestic economy to grow at the rate n, and the foreign economy to grow at n^*. Consider (3.33) and assume that the trade balance grows at the domestic growth rate. Alternatively, consider (3.37) and assume that each country's foreign assets per GDP remains stable. Both approaches immediately result in:

$$\frac{p(\mathfrak{t})}{p(\mathfrak{t}+1)} = \frac{1+n^*}{1+n} \quad \text{if} \quad \text{л} \neq 0 \quad \forall\, \mathfrak{t} \tag{4.20}$$

This immediately leads to our first result. There can be no steady state with an imbalance in the rate of growth unless there is a continuous change in the terms of trade. Since the domestic economy is growing faster, its terms of trade deteriorate. The domestic commodity becomes cheaper to ensure that the share of output of the domestic commodity in word output has not changed. It is surprising that this result is not more widely known.[12] It is the key relationship that enables differential growth in the steady state of the economy. Some conceptual difficulty lies in the fact that the terms of trade do not stay stable in this steady state.

Using (4.20) in (3.18) (with $\mathfrak{t}' = \mathfrak{t}+1$), I get

$$(1+r) = (1+\text{д})\,(1+n)^{\sigma-(1-\sigma)\,(1-\alpha)}\,(1+n^*)^{(1-\sigma)\,(1-\alpha)} \tag{4.21}$$

In the (n, r) plane, this expression defines the intertemporal "demand" curve of Krichel and Levine (1996), the KL curve. Since $\sigma > 0$, this curve is upward sloping. The impact of the foreign growth rate on domestic growth depends crucially on the elasticity of substitution. In the standard case where the elasticity is smaller than one[13] we have a negative impact of the foreign growth on domestic growth. Any increase in the foreign growth rate will shift the domestic KL curve upwards in the (n, r) plane. When the elasticity is high, $\sigma < 1$, and the impact of the foreign growth rate on the domestic growth rate is positive. Note that when the growth rates in both countries are equal, then (4.21) simplifies to (4.12). The KL curve for the foreign economy is

$$(1+r^*) = (1+\text{д}^*)\,(1+n^*)^{\sigma^*+(1-\sigma^*)\,(1-\alpha^*)}\,(1+n)^{(1-\sigma^*)\,(1-\alpha^*)} \tag{4.22}$$

[12] I believe that this may be the first time this result has been uncovered but it is difficult to verify that. The claim may have been made earlier, but not been published since a previous researcher may have thought that this result was not worth mentioning.

[13] The received wisdom σ is 2.

But note that (3.38) requires that in the steady state

$$\frac{1+r^*}{1+r} = \frac{1+n^*}{1+n} \tag{4.23}$$

Dividing (4.21) by (4.22) and making use of (4.23), we get

$$\frac{(1+n)^{\varphi}}{(1+n^*)^{\varphi^*}} = \frac{1+\text{д}}{1+\text{д}^*} \tag{4.24}$$

where $\varphi = \alpha(1-\sigma) - (1-\sigma^*)(1-\alpha^*)$ and $\varphi^* = \alpha^*(1-\sigma^*) - (1-\sigma)(1-\alpha)$. (4.24) must hold as an identity[14] for *all* n^* and n. Otherwise the system would be overdetermined. In a conventional neoclassical growth model, $n = n^* = 0$ and the condition (4.24) reduces to the well-known requirement that the discount rates in the two economies must be equal for an international steady state with perfect capital markets to exist[15]. If growth is *symmetric*, then in addition we require that $\varphi = \varphi^*$. It is straightforward to see that this requires that $\sigma = \sigma^*$. If there is a growth differential we require the more stringent condition that $\varphi = \varphi^* = 0$. The equality to zero requires that $\alpha + \alpha^* = 1$, in addition to the previous conditions. We can summarize

Proposition 4.2 There is no steady state in the model unless $\text{д} = \text{д}^*$. There is no steady state with positive growth unless $\text{д} = \text{д}^*$. There is no steady state with different growth rates unless $\alpha + \alpha^* = 1$.

The KL curves of (4.21) and (4.22) can be thought of as intertemporal demand curves. For each country, a supply curve can be derived from (4.6). Using (4.10) for the private capital stock and (4.8) for the public capital stock, we obtain

$$\delta + n = \bar{\epsilon}^{1/(1-\gamma)} \left(\frac{(1-\tau)\,\kappa}{r+\delta} \right)^{\gamma/(1-\gamma)} \tau\,\text{г} \tag{4.25}$$

This relationship also holds for the foreign economy. Since these curves are dependent on the Cobb-Douglas production function, I will label them the "CD curves". They are downward sloping in the (n, r) space. Since the KL curves (4.21) and (4.22) are upward sloping, a unique equilibrium will exist in each economy. This equilibrium is parameterized by the foreign growth rate.

It is interesting to note that the growth rate in the steady state can be related to exogenous parameters and policy variables. It does not depend on foreign assets. When a country has foreign assets, it will be able to have an expenditure that exceeds income. Consumption, but not investment will depend on the foreign assets

$$c = (1-\gamma)(1-\tau) + \frac{\gamma(1-\tau)(r-n)}{r+\delta} + \frac{(r-n)\,\text{л}}{(1+n)}$$

[14] This becomes clear when looking at equation (4.25) and its foreign counterpart (not reproduced). These two equations write the domestic and foreign interest rate as functions of the domestic and foreign growth rate, respectively. Thus the system (4.21), (4.22) and (4.23) has only two unknowns, n and n^*.

[15] Of course this requirement only holds in models with a homogeneous infinitely lived private sector, see Buiter (1981).

$\mathcal{A}$ is an exogenous variable here. Growth rates and interest rates can not be used to explain foreign asset accumulation. Note that this a general characteristic of models with infinite lives. In these models any long run accumulation of foreign assets is possible, as long as it respects the positivity of consumption in all countries.

It should also be noted that when searching to maximise the domestic growth rate, there is no need to take account of the other country's. Differentiate (4.25) and (4.21)

$$\mathrm{d}n = \frac{\delta + n}{\tau}\,\mathrm{d}\tau - \frac{\gamma}{1-\gamma}\frac{\delta + n}{1-\tau}\,\mathrm{d}\tau \tag{4.26}$$

The maximisation of growth is reached when $\tau = 1 - \gamma$, thus is is independent from the foreign growth rate. This is a result that generalises from Devereux and Mansoorian (1992).

There are two interpretations for the Subsection. On the one hand, I have shown that a situation exists where the domestic and foreign growth rates are different forever. To enable such a scenario, I need to ensure that the size of the slow-growing economy is not zero in the long run. I show that with a permanent change in the terms of trade, this is indeed possible. In this situation, the domestic output will grow at the domestic growth rate if expressed in units of the domestic commodity, and it will grow at the foreign growth rate if expressed in units of the foreign commodity. Under those circumstances, it does not matter for a consumer if output grows at a slow or a fast rate, since all the benefits of high growth in the domestic commodity output are lost through the decline in the terms of trade. Growth does not matter.

An alternative view of my findings comes from the impact on domestic growth of a change in foreign growth. I have shown that for common parameter values, an increase in foreign growth reduces domestic growth. This is quite a general result. Assume that the consumer's intertemporal substitution is smaller than one, i.e. that a percentage change in relative prices leads to a smaller change in consumption growth. In this realistic case there is a positive long-run relationship between interest rates and growth rates. It does matter little whether it is domestic or foreign growth I am referring to. Any increase in the growth rate will lead to an increase in the required interest rate. An increase in the interest rate depresses private capital accumulation. A reduction in capital accumulation reduces the rate of growth. Therefore an increase of growth in one country raises the world interest rate, and depresses growth in another country. These relationships hold under very general conditions.

All my findings depend crucially on the assumption that the demand shares α and α^* remain fixed. That implies that the consumption demand addressed to domestic producers always has a fixed share in world consumption. This is not realistic when growth is driven by the accumulation of knowledge. In that case the share of each country in the world economy is endogenous. There is a large volume of such models, but they all share a logarithmic felicity function that rules out the growth externalities that I have considered in this paper. Bringing together both strands is a challenge that has yet to be taken up, but I am convinced that it will lead to further insight into the process of relative development of different countries.

5 Linear-quadratic approximation and calibration

In order to study the dynamics of the model I develop a linear-quadratic approximation of the model. This approximation is a necessary step to apply the solution procedures outlined in Appendix A and B. I am not aware of any other technique that will compute a time-consistent trajectory[16]. Unfortunately it is common practice to introduce large direct penalties on the use of instruments in this type of exercise. On many an occasion they need to be introduced for the model to give reasonable results. The origin of the problems lies in the linear nature of the underlying set of constraints and the crude approximation that the quadratic form allows for and lack of care in the modeling process. An important feature of this thesis is to show that when great care is taken in the approximation of the target function however, then the linear-quadratic approach can yield equilibrium values that have plausible magnitudes, without having to resort to large penalties of the objectives.

5.1 Linearisation

I linearise about a symmetric steady state, using the notation $x_t = x(\mathrm{t}) - x$ where x is the steady state of a variable. To conserve space, I first write the equations that hold for both economies in the sense that the equations for the foreign equations are simply "starred" version of the domestic equations.

$$\begin{aligned} c_t = {} & \frac{1+r}{(1+\xi)\,(1+n)}\, w_{t-1} + \frac{w}{(1+\xi)\,(1+n)}\, r_{t-1} \\ & - \frac{w\,(1+r)}{(1+\xi)^2\,(1+n)}\, \xi_t - \frac{w\,(1+r)}{(1+\xi)\,(1+n)^2}\, n_t \end{aligned} \tag{5.1}$$

$$k_t = \frac{1-\delta}{1+n}\, k_{t-1} - \frac{(1-\delta)\,k}{(1+n)^2}\, n_t + i_t \tag{5.2}$$

$$\begin{aligned} r_t = {} & -\bar{\epsilon}\,\gamma\, k^{\gamma-1}\, k^{g\,1-\gamma} \tau_{t+1} - \bar{\epsilon}\,(1-\tau)\,\gamma\,(1-\gamma)\, k^{\gamma-2}\, k^{g\,1-\gamma}\, k_t \\ & + \bar{\epsilon}\,(1-\tau)\,\gamma\,(1-\gamma)\, k^{\gamma-1}\, k^{g\,-\gamma}\, k^g_t \end{aligned} \tag{5.3}$$

$$\frac{n_{t+1}}{1+n} = \frac{\gamma}{k}\, k_t + \frac{1-\gamma}{k^g}\, k^g_t \tag{5.4}$$

$$k^g_t = \frac{1-\delta}{1+n}\, k^g_{t-1} - \frac{(1-\delta)\,k^g}{(1+n)^2}\, n_t + g^i_t \tag{5.5}$$

$$g^i_t = \tau\, \mathrm{r}_t + \mathrm{r}\, \tau_t$$

$$w^h_t = \frac{w^h}{1+n}\, r_{t-1} + \frac{1+r}{1+n}\, w^h_{t-1} - \frac{(1+r)\, w^h}{(1+n)^2}\, n_t - (1-\kappa)\, \tau_t$$

$$g^c_t = (1-\mathrm{r})\, \tau_t - \tau\, \mathrm{r}_t \tag{5.6}$$

[16] There is an old mimeo by Albert Marcet on that topic, but it was never formally published. It is basically incomprehensible, but for what I understand of it, it can not deal with a model like ours

The following equations only hold for each country separately

$$
\begin{aligned}
a_t &= \frac{1+r}{1+n}\, a_{t-1} - \frac{a\,(1+r)}{(1+n)^2}\, n_t + \frac{a}{1+n}\, r_{t-1} - \tau_t - i_t - c_t && (5.7)\\
\xi_{t+1} &= \frac{1+\xi}{\xi}\, \xi_t + \frac{(1-\sigma)\,(1-\alpha)}{\sigma}\, \frac{1+\xi}{p}\, p_{t+1} \\
&\quad - \frac{(1-\sigma)\,(1-\alpha)}{\sigma}\, \frac{1+\xi}{p}\, p_t + \frac{(1+\xi)\,(1-1/\sigma)}{1+r}\, r_t && (5.8)\\
\xi^*_{t+1} &= \frac{1+\xi}{\xi}\, \xi^*_t - \frac{(1-\sigma)\,(1-\alpha)}{\sigma}\, \frac{1+\xi}{p}\, p_{t+1} \\
&\quad + \frac{(1-\sigma)\,(1-\alpha)}{\sigma}\, \frac{1+\xi}{p}\, p_t + \frac{(1+\xi)\,(1-1/\sigma)}{1+r}\, r^*_t && (5.9)\\
p_{t+1} &= p_t + \frac{p}{1+r}\, r_t - \frac{p}{1+r^*}\, r^*_t \\
i_t &= -\tau_t - \alpha\, c_t - (1-\alpha)\, p\, c^*_t - (1-\alpha)\, c\, p_t && (5.10)\\
i^*_t &= -\tau^*_t - \alpha\, c^*_t - \frac{(1-\alpha)}{p}\, c_t + \frac{(1-\alpha)\, c}{p^2}\, p_t \\
w_{t-1} &= k_{t-1} + w^{h}_{t-1} + a_{t-1} && (5.11)\\
w^*_{t-1} &= k^*_{t-1} + w^{h*}_{t-1} - \frac{a_{t-1}}{p} + \frac{a}{p^2}\, p_{t-1} && (5.12)
\end{aligned}
$$

5.2 The welfare function

Abstracting for the moment from the fact that I have a two-country model with government spending, consider the target function

$$U(t) = \sum_{t'=1}^{\infty} \varrho^{t'-1}\, \frac{C(t')^{1-\sigma} - 1}{1-\sigma} \tag{5.13}$$

Introducing $c(t) = C(t)/Q(t)$, $n(t) = (Q(t) - Q(t-1))/Q(t-1)$ and removing constant terms, maximising (5.13) is equivalent to maximising

$$U(t) = \sum_{t'=1}^{\infty} \varrho^{t'-1}\, \frac{\left[c(t)\, \prod_{t''=1}^{t'} (1 + n(t''))\right]^{1-\sigma}}{1-\sigma} \tag{5.14}$$

Now defining a modified discount rate

$$\mathring{\varrho} = \varrho\, (1+n)^{1-\sigma} \tag{5.15}$$

and introducing the notation

$$n^b_t = n^b_{t-1} + n_t \tag{5.16}$$

the quadratic approximation of (5.14) is given by[17]

$$U_{\mathrm{t}} = \sum_{\mathrm{t}'=1}^{\infty} \mathring{\varrho}^{\mathrm{t}'-1}\, u_{\mathrm{t}'} \tag{5.17}$$

where

$$\begin{aligned} u_{\mathrm{t}} = {} & c^{-\sigma}\,(1+n)^{1-\sigma}\, c_{\mathrm{t}} + \frac{\mathring{\varrho}}{1-\mathring{\varrho}}\, c^{1-\sigma}\,(1+n)^{-\sigma}\, n_{\mathrm{t}} \\ & - \sigma\, c^{-\sigma-1}\,(1+n)^{1-\sigma}\, \frac{c_{\mathrm{t}}^2}{2} - \frac{\sigma\,\mathring{\varrho}}{1-\mathring{\varrho}}\, c^{1-\sigma}\,(1+n)^{-1-\sigma}\, \frac{n_{\mathrm{t}}^2}{2} \\ & + \frac{1-\sigma}{c^{\sigma}\,(1+n)^{\sigma}}\, n_{\mathrm{t}}^{\mathrm{b}}\, c_{\mathrm{t}} + \frac{\mathring{\varrho}\,(1-\sigma)\, c^{1-\sigma}}{(1-\mathring{\varrho})\,(1+n)^{1+\sigma}}\, n_{\mathrm{t}}\, n_{\mathrm{t}-1}^{\mathrm{b}} \end{aligned} \tag{5.18}$$

Integrating government consumption is straightforward since it is added with a coefficient υ_c. However simulations based on (5.18) with added government consumption terms show that the resulting function is not accurate enough for the calculations to work. In the precommitment case, the calculation of the Riccati (A.17) of page 102 fails. The procedure of Subsection A.3 does not converge either.

A more accurate approximation of the welfare criterion can be achieved by replacing the *linear* terms in c_{t}, $g_{\mathrm{t}}^{\mathrm{c}}$ and n_{t}, by their quadratic approximations gained from expanding (5.1), (5.4) and (5.6) to a further degree. Adding all these components

[17] This can be seen when writing out the first few terms of the sum.

$$\begin{aligned} U_{\mathrm{t}} = {} & c^{-\sigma}\,(1+n)^{1-\sigma}\, c_{\mathrm{t}} + c^{1-\sigma}\,(1+n)^{-\sigma}\, n_{\mathrm{t}}^{\mathrm{b}} - \sigma\, c^{-1-\sigma}\,(1+n)^{1-\sigma}\, c_{\mathrm{t}}^2/2 \\ & + (1-\sigma)\, c^{-\sigma}\,(1+n)^{-\sigma}\, c_{\mathrm{t}}\, n_{\mathrm{t}} - \sigma\, c^{1-\sigma}\,(1+n)^{-1-\sigma}\, n_{\mathrm{t}}^2/2 + \mathring{\varrho}\big[c^{-\sigma}\,(1+n)^{(1-\sigma)}\, c_{\mathrm{t}+1} \\ & + c^{1-\sigma}\,(1+n)^{(1-\sigma)-1}\, n_{\mathrm{t}+1}^{\mathrm{b}} - \sigma\, c^{-1-\sigma}\,(1+n)^{(1-\sigma)}\, c_{\mathrm{t}+1}^2/2 \\ & + (1-\sigma)\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+1}\, n_{\mathrm{t}}^{\mathrm{b}} + (1-\sigma)\, c^{-\sigma}\,(1+n)^{(1-\sigma)-1}\, c_{\mathrm{t}+1}\, n_{\mathrm{t}+1}^{\mathrm{b}} \\ & - \sigma\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}}^2/2 - \sigma\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+1}^2/2\big] \\ & + \mathring{\varrho}^2\big[c^{-\sigma}\,(1+n)^{(1-\sigma)}\, c_{\mathrm{t}+2} + c^{1-\sigma}\,(1+n)^{(1-\sigma)-1}\, n_{\mathrm{t}+2}^{\mathrm{b}} \\ & - \sigma\, c^{-1-\sigma}\,(1+n)^{(1-\sigma)-2}\, c_{\mathrm{t}+2}^2/2 + (1-\sigma)\, c^{-\sigma}\,(1+n)^{(1-\sigma)-1}\, c_{\mathrm{t}+2}\, n_{\mathrm{t}+2}^{\mathrm{b}} \\ & + (1-\sigma)\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+1}\, n_{\mathrm{t}} + (1-\sigma)\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+2}\, n_{\mathrm{t}+1}^{\mathrm{b}} \\ & - \sigma\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}}^2/2 - \sigma\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+1}^2/2 \\ & - \sigma\, c^{1-\sigma}\,(1+n)^{(1-\sigma)-2}\, n_{\mathrm{t}+2}^2/2\big] \ldots \end{aligned}$$

and dividing through $(1+n)^{1-\sigma}$ gives

$$
\begin{aligned}
u_{\mathrm{t}} = {} & c^{-\sigma}\, c_{\mathrm{t}} + \frac{c^{-\sigma}\, w\,(1+r)}{(1+n)^3\,(1+\xi)}\, n_{\mathrm{t}}^2 \\
& + \frac{c^{-\sigma}\, w\,(1+r)}{(1+n)\,(1+\xi)^3}\, \xi_{\mathrm{t}}^2 + \frac{c^{-\sigma}}{(1+\xi)\,(1+n)}\, w_{\mathrm{t}-1}\, r_{\mathrm{t}-1} \\
& - \frac{c^{-\sigma}\,(1+r)}{(1+\xi)^2\,(1+n)}\, w_{\mathrm{t}-1}\, \xi_{\mathrm{t}} - \frac{c^{-\sigma}\,(1+r)}{(1+\xi)\,(1+n)^2}\, w_{\mathrm{t}-1}\, n_{\mathrm{t}} \\
& - \frac{c^{-\sigma}\, w}{(1+\xi)^2\,(1+n)}\, r_{\mathrm{t}-1}\, \xi_{\mathrm{t}} - \frac{c^{-\sigma}\, w}{(1+\xi)\,(1+n)^2}\, r_{\mathrm{t}-1}\, n_{\mathrm{t}} \\
& + \frac{c^{-\sigma}\,(1+r)\, w}{(1+\xi)^2\,(1+n)^2}\, \xi_{\mathrm{t}}\, n_{\mathrm{t}} + \upsilon_c\, g^{\mathrm{c}-\sigma}\, g^{\mathrm{c}}_{\mathrm{t}} \\
& - \upsilon_c\, g^{\mathrm{c}-\sigma}\, \tau_{\mathrm{t}}\, \mathrm{r}_{\mathrm{t}} + \left[c^{1-\sigma} + \upsilon_c\, g^{\mathrm{c}1-\sigma}\right] \frac{\mathring{\varrho}}{1-\mathring{\varrho}}\, \frac{n_{\mathrm{t}}}{1+n} \\
& - \left[c^{1-\sigma} + \upsilon_c\, g^{\mathrm{c}1-\sigma}\right] \frac{(1-\gamma)\,\gamma\,\mathring{\varrho}}{2\,(1-\mathring{\varrho})} \left[\frac{k_{\mathrm{t}-1}}{k} - \frac{k^{\mathrm{g}}_{\mathrm{t}-1}}{k^{\mathrm{g}}}\right]^2 \\
& - \sigma\, c^{-\sigma-1}\, \frac{c_{\mathrm{t}}^2}{2} - \sigma\, \upsilon_c\, g^{\mathrm{c}-\sigma-1}\, \frac{g^{\mathrm{c}2}_{\mathrm{t}}}{2} \\
& - \frac{\sigma\,\mathring{\varrho}}{1-\mathring{\varrho}} \left[c^{1-\sigma} + \upsilon_c\, g^{\mathrm{c}1-\sigma}\right] \frac{n_{\mathrm{t}}^2}{2\,(1+n)^2} \\
& + (1-\sigma)\, \frac{c^{-\sigma}}{1+n}\, n^{\mathrm{b}}_{\mathrm{t}}\, c_{\mathrm{t}} + (1-\sigma)\, \frac{\upsilon_c\, g^{\mathrm{c}-\sigma}}{1+n}\, n^{\mathrm{b}}_{\mathrm{t}}\, g^{\mathrm{c}}_{\mathrm{t}} \\
& + \frac{(1-\sigma)\,\mathring{\varrho}}{(1-\mathring{\varrho})\,(1+n)^2} \left[\upsilon_c\, g^{\mathrm{c}1-\sigma} + c^{1-\sigma}\right] n_{\mathrm{t}}\, n^{\mathrm{b}}_{\mathrm{t}-1}
\end{aligned}
\tag{5.19}
$$

For the two-country model I use the consumption expenditure approach. Recall that

$$
\begin{aligned}
c_{\mathrm{d}}(\mathrm{t}) &= \alpha\, c(\mathrm{t}) \\
c_{\mathrm{f}}(\mathrm{t}) &= \frac{1-\alpha}{p(\mathrm{t})}\, c(\mathrm{t})
\end{aligned}
\tag{5.20}
$$

Thus the term $c(\mathrm{t})$ of (5.14) according to (3.2) corresponds to the expression $\alpha^{\alpha}\,(1-\alpha)^{\alpha-1}\, c(\mathrm{t})\, p(\mathrm{t})^{1-\alpha}$. Therefore the equivalent to the maximisation of (3.1) is the maximisation of

$$
U(\mathrm{t}) = \sum_{\mathrm{t}'=1}^{\infty} \varrho^{\mathrm{t}'-1} \frac{\left[p(\mathrm{t})^{\alpha-1}\, c(\mathrm{t})\, \prod_{\mathrm{t}''=1}^{\mathrm{t}'} (1+n(\mathrm{t}'))\right]^{1-\sigma}}{1-\sigma}
\tag{5.21}
$$

This implies that in the two-country case, (5.19) is replaced by

$$
\begin{aligned}
u_{t} = {} & p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\, c_{t} + \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\, w\,(1+r)}{(1+n)^{3}\,(1+\xi)}\, n_{t}^{2} \\
& + \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\, w\,(1+r)}{(1+n)\,(1+\xi)^{3}}\, \xi_{t}^{2} + \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}}{(1+\xi)\,(1+n)}\, w_{t-1}\, r_{t-1} \\
& - \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\,(1+r)}{(1+\xi)^{2}\,(1+n)}\, w_{t-1}\, \xi_{t} - \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\,(1+r)}{(1+\xi)\,(1+n)^{2}}\, w_{t-1}\, n_{t} \\
& - \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\, w}{(1+\xi)^{2}\,(1+n)}\, r_{t-1}\, \xi_{t} - \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\, w}{(1+\xi)\,(1+n)^{2}}\, r_{t-1}\, n_{t} \\
& + \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}\,(1+r)\, w}{(1+\xi)^{2}\,(1+n)^{2}}\, \xi_{t}\, n_{t} + \upsilon_{c}\, g^{c-\sigma}\, g_{t}^{c} \\
& - \upsilon_{c}\, g^{c-\sigma}\, \tau_{t}\, \Gamma_{t} + \left[p^{(\alpha-1)(1-\sigma)}\, c^{1-\sigma} + \upsilon_{c}\, g^{c1-\sigma}\right] \frac{\mathring{\varrho}}{1-\mathring{\varrho}}\, \frac{n_{t}}{1+n} \\
& - \left[p^{(\alpha-1)(1-\sigma)}\, c^{1-\sigma} + \upsilon_{c}\, g^{c1-\sigma}\right] \frac{(1-\gamma)\,\gamma\,\mathring{\varrho}}{2\,(1-\mathring{\varrho})} \left[\frac{k_{t-1}}{k} - \frac{k_{t-1}^{g}}{k^{g}}\right]^{2} \\
& - \sigma\, p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma-1}\, \frac{c_{t}^{2}}{2} - \sigma\, \upsilon_{c}\, g^{c-\sigma-1}\, \frac{g_{t}^{c2}}{2} \\
& - \frac{\sigma\,\mathring{\varrho}}{1-\mathring{\varrho}} \left[p^{(\alpha-1)(1-\sigma)} c^{1-\sigma} + \upsilon_{c}\, g^{c1-\sigma}\right] \frac{n_{t}^{2}}{2\,(1+n)^{2}} \\
& + (1-\sigma)\, \frac{p^{(\alpha-1)(1-\sigma)}\, c^{-\sigma}}{1+n}\, n_{t}^{b}\, c_{t} + (1-\sigma)\, \frac{\upsilon_{c}\, g^{c-\sigma}}{1+n}\, n_{t}^{b}\, g_{t}^{c} \\
& + \frac{(1-\sigma)\,\mathring{\varrho}}{(1-\mathring{\varrho})\,(1+n)^{2}} \left[p^{(\alpha-1)(1-\sigma)}\, c^{1-\sigma} + \upsilon_{c}\, g^{c1-\sigma}\right] n_{t}\, n_{t-1}^{b} \\
& + (\alpha-1)\,(1-\sigma)\, p^{(\alpha-1)(1-\sigma)-1}\, c^{1-\sigma}\, p_{t} \\
& + (\alpha-1)\,(1-\sigma)\, \frac{p^{(\alpha-1)(1-\sigma)-1}\, c^{1-\sigma}}{1+n}\, n_{t}^{b}\, p_{t} \\
& + (\alpha-1)\,(1-\sigma)\,((\alpha-1)\,(1-\sigma)-1)\, p^{(\alpha-1)(1-\sigma)-2}\, c^{1-\sigma}\, \frac{p_{t}^{2}}{2} \\
& + (\alpha-1)\,(1-\sigma)^{2}\, p^{(\alpha-1)(1-\sigma)-1} c^{-\sigma}\, c_{t}\, p_{t}
\end{aligned}
\tag{5.22}
$$

Note that the model would not converge with a more simple target!

All results in Sections 6, 7 and 8 were computed using the ACES package, see Gaines, al Nowaihi, and Levine (1989). This is a library that can be used on any F77 compiler in conjunction with the NAG numerical routines. The programs used to compute the results as well as the ACES libraries are available on request from the author. NAG is a commercial programme and needs to be obtained separately, but it is very widely used.

In actual calculations there are two adjustments to the theoretical setup developed in this section. An interesting problem arises from the definition of n_{t}^{b}. In an economy with perpetual growth, there is no steady state to this variable. This variable must be included in the state vector. But the procedure that calculates the time-consistent strategy will look for a solution with a stable state vector. Therefore in the time-consistent solution, there will be no long run change of growth if I introduce n^{b} as in (5.16). I will be referring to this effect as the "accumulator effect" in the following. The accumulator effect on the growth rate implies the absence of a change in long run growth. Through the logic of the linear representation of the model, it also implies that the instruments are set to the initial steady-state values in the long run, i.e. $\tau_\infty = \mathrm{r}_\infty = 0$. Thus from any steady state, the system under a time-consistent policy would be going back to the steady state where it started from, which is not a very satisfactory feature. To circumvent the problem, it would suffice to reduced the root below 1 by a very small amount. In practice it turns out that if the root is too close to 1, the stationary requirement still has an important negative impact on long run growth. The results of the time consistent calculations are convergent in the sense that as the root approaches 1, the long run rate of growth change approaches 0. After a large number of experiments, I set the root to .99. Note that the variable n^{b} is only used in the specification of the government's objective and it is not used anywhere else.

A second case where the calculations diverge from the theory that I set out here is the presence of a startup penalty. Within the ACES software, it is not possible to specify inequality constraints on the behaviour of the system. Thus it is not possible to state that in fact, consumption must always be positive and that the tax must lie between zero and one. Thus, in the case of the optimal policy, optimisation in date 1 implies setting the tax to infinity and satisfy all spending from the revenue generated in the first period. To exclude that problem I need to install a very small penalty on the instruments. These can vary from one regime to another and are discussed with the description of the results. All are so small that their removal for the calculation of the time-consistent regime makes no change in the results for these regimes.

Note that the first problem only affects the time-consistent solution, whereas the second problem only affects the optimal control calculation. However I kept the same programme for both calculations, unless otherwise stated in the results. In every other respect the optimisation routines exactly replicate the objectives as set out here.[18] They do converge and the results are meaningful. This shows that if the linear-quadratic approximation is being prepared in a very careful way, then it is possible to obtain meaningful results out of the numerical exercises without having to resort to heavily penalising instrument changes.

5.3 The calibration

There are two approaches towards calibrating a model. The first consists in collecting data about observable variables like consumption, investment, growth, etc, and deduce variables that are not observed from the steady state of the model. This approach is simple and intuitive. A second approach would do the opposite, i.e. use different scenarios of the unobserved variables to see whether in the steady state these will give values for the observed variable that conform to observation. This method has the advantage of allowing for "what if" simulations to study the effect

[18] The programmes are available on request.

	Category	A	B	C
0	Total expenditure	1194.60	501.66	438.77
1	General public services	78.71	14.25	23.63
2	Defense	293.54		
3	Public order and safety	10.57	20.01	40.83
4	Education	21.50	169.14	191.57
5	Health	154.19	87.30	36.27
6	Social security & welfare	317.82	82.74	32.39
7	Housing and community amenities	32.62	3.61	13.01
8	Recreation, cultural & religious affairs	3.21	2.67	13.69
9	Fuel & energy	5.41	.26	2.14
10	Agriculture, forestry, fishing & hunting	21.89	8.99	2.42
11	Mining, manufacturing & construction	.64		
12	Transports & communications	29.89	46.74	27.23
13	Other economic affairs & services	38.29	7.04	2.20
14	Other expenditures	187.17	58.91	53.39

Table 5.1: 1990 US government expenditure. Column A is "consolidated central government" expenditure, B is "state region and province government", and C is "local government" (source: IMF Government Finance Statistics Yearbook)

of changes in the unobserved exogenous parameters. I adopt a mixed approach here. The main source for parameter values is Chari, Jones, and Manuelli (1994). They set $\tau = 22\%$, $\varrho = 98\%$ (which corresponds to д $\approx 2\%$), $\sigma = 2.0$, $\kappa = 36\%$, $n = 2\%$ and $\delta = 4\%$. In addition we need two more parameters that are г, and γ. The latter is difficult to quantify, but a reasonable baseline value should be the ratio of private capital in the total capital stock, i.e. we use $\gamma = k/(k + k^{\mathrm{g}})$. To obtain an estimate for г I collected data for various categories of US government expenditure numbered 1–14 in Table 5.1. I assume that the categories 4, 5 and 12 are the expenditures contributing to the capital stock of the government. I can then compute г, the proportion of investment expenditure, as г $\approx 36\%$. The remaining parameter values are derived using the steady state relationships as set out in Table 5.2. The alert reader will have noticed that Table 5.2 only calibrates the single-country version of the model. The calibration for two countries is based on two single countries. I set $\alpha = .5$ and л $= 0$ and think of the world as consisting of two economies with the same characteristics. It would of course be interesting to calibrate two asymmetric economies, in particular two economies at a different rate of long run growth. One could then examine the success of various regimes in closing the growth gap. However I have not found a state-space representation of such a model.

$$
\begin{aligned}
g^{\mathrm{c}} &= \tau - \tau\,\text{г} && \approx 0.141 \\
g^{\mathrm{i}} &= \tau\,\text{г} && \approx 0.079 \\
k^{\mathrm{g}} &= \frac{(1+n)\,g^{\mathrm{i}}}{n+\delta} && \approx 1.346 \\
r &= (1+\text{д})\,(1+n)^{\sigma} - 1 && \approx 0.061 \\
\xi &= \frac{1}{(1+r)^{\sigma-1}\,(1+\text{д})^{-\sigma}} && \approx 24.75 \\
k &= \frac{\kappa\,(1-\tau)\,(1+n)}{\delta+r} && \approx 2.830 \\
i &= k\,\frac{n+\delta}{1+n} && \approx 0.166 \\
\text{щ}^{\mathrm{h}} &= \frac{(1+n)\,(1-\tau)\,(1-\kappa)}{r-n} && \approx 12.36 \\
\text{щ} &= k + \text{щ}^{\mathrm{h}} && \approx 15.19 \\
c &= \frac{(1+r)\,\text{щ}}{(1+n)\,(1+\xi)} && \approx 0.614 \\
\bar{\epsilon} &= \frac{1+n}{k^{\gamma} k^{\mathrm{g}\,1-\gamma}} && \approx 0.458 \\
\gamma &= \frac{k}{k^{\mathrm{g}}+k} && \approx 0.678
\end{aligned}
$$

Table 5.2: Calibration where n, σ, д, κ, δ, γ, г and τ are fundamental

6 Shocks in the Linear Model

The comparative static analysis based on the steady state has the advantage that it uses the full model. But it does not take into account any transitional dynamics between one state and an alternative state. This transition is discussed here in the context of the linearized from of the model, equations (5.1) to (5.12). I examine the impact of changes in the policy variable and the transition to a new steady state. To compare the open economy with the case of a closed economy, I compare the effect of a .5% change in a policy variable in the closed economy with a 1% change in the Home economy only. From Figure 6.3 onwards, I represent the domestic variable by a $\Diamond$, the foreign variable with a $\Box$, and the reference value in a single country that corresponds to a shock of half the size of the shock in the two-country world with a $+$. Correspondingly, if x is a variable of the model, then I will denote x_t the deviation from the steady state of the domestic value, x_t^* the deviation from the steady state of the foreign value, and $\tilde{x}_t$ the deviation from the steady state of both domestic and foreign value, for a shock of half the impact in both countries. All figures on the second axis of every graph are percentages.

All shocks in this section are permanent,[19] because I wish to examine the properties of the long-run steady state with growth differentials. Assuming that there is a shift from symmetric to asymmetric policies, I am interested to find out what happens to the terms of trade and to foreign assets. Note that there are no implications for the government budget because all are balanced-budget changes. There are two policy variables in the model, the tax rate τ and the fraction of expenditure devoted to infrastructure r. A shock to the latter is more straightforward to understand than a shock to the former. When considering a change in economic

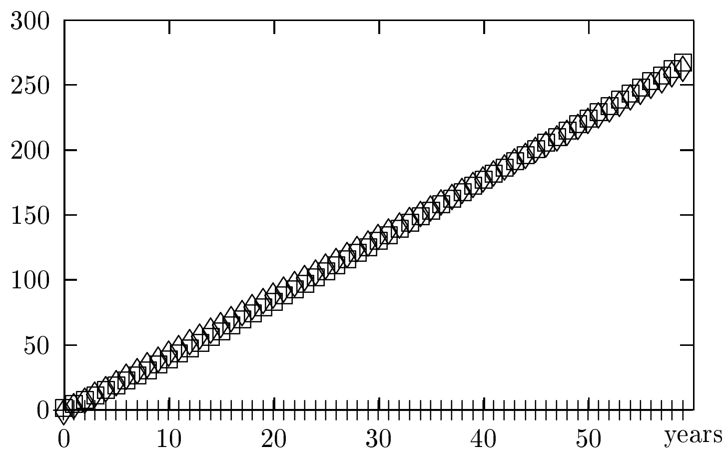

Figure 6.1: Foreign assets $\mathcal{A}_t$, r shock $\leftrightsquigarrow$ $\Diamond$, τ shock $\leftrightsquigarrow$ $\Box$

policy in the domestic country only, the question of asymmetric growth immediately

[19]In an endogenous growth model like ours, any temporary change in policy would lead to a permanent change in the long-run growth. Jones (1995) and Kocherlakota and Yi (1996) have used that fact to test if the time series of US output have that property.

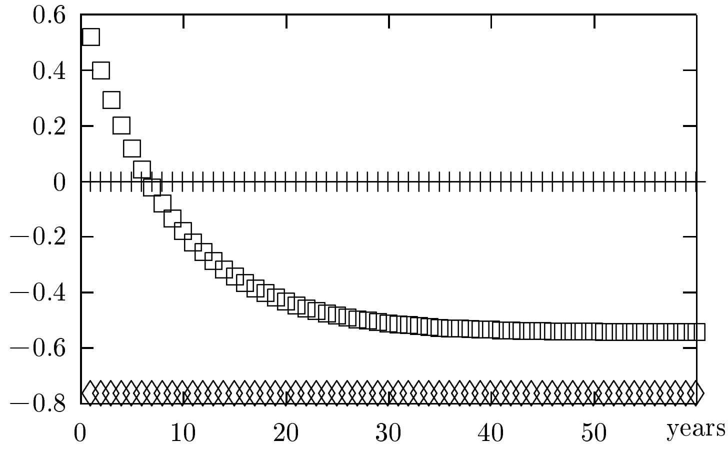

Figure 6.2: Terms of trade p_t, r shock $\leadsto \Diamond$, τ shock $\leadsto \Box$

arises, because there is a unique mapping from the policy instruments to the interest and growth rates. From the results in Section 4, one would expect that the relative price rises continuously in order to make up for the differential in growth rates. For example we would expect that $\lim_{t\to\infty} p_t = \infty$, when $\mathrm{r}_t = 1 \; \forall \, t \geq 1$. But this is not the case here. In fact the results of the simulation are that when differential growth arises, we have a continuous adjustment of the foreign assets

$$\mathcal{A}_t = \mathcal{A}_{t-1} + \not{c} \text{ when } \mathrm{r}_t = 1 \qquad \forall \, t \geq 1 \tag{6.1}$$

where $\not{c}$ is general placeholder for a constant. This is pictured as the $\Diamond$ line in Figure 6.1. Figure 6.2 introduces another striking feature of the solution

$$p_t = \not{c} \text{ when } \mathrm{r}_t = 1, \qquad \forall \, t \geq 1 \tag{6.2}$$

The shock on r produces a once-and-for-all *fall* of relative price, a change in the opposite direction then the one predicted by the steady-state analysis. Thus the results that I find here are the opposite of the results in Section 4. Remember that in the comparative statics comparison, the terms of trade are subject to continuous change and the foreign assets remain unchanged. Here the terms of trade remain constant and the foreign assets accumulate. There is no steady state for foreign assets.

The only common element with the earlier results are that both steady states are incomplete, in the sense that there is a subset of variable that continues to change, whereas another set of variables remain constant over time. Despite the continuous rise of foreign assets, most other variables are stable. Thus we do not reach a steady state in all variables, but a "semi" steady state, where only a large subset of the variables remains constant.

To understand that these results nevertheless makes sense, it is instructive to look at diagrams that trace the evolution of selected variables over time. In Subsection 6.1 I discuss the infrastructure shock, and in Subsection 6.2 the shock to taxation.

6.1 A shock to infrastructure expenditure

When the government increases the fraction allocated to infrastructure, say $\mathrm{r}_t = 1\%$ $\forall\, t \geq 1$, then government investment is increased, and government consumption falls by the same amount. Thus there is no change in government spending and no impact on taxation. This greatly simplifies the problem under study. In addition, the felicity function (3.2) implies that there is no direct substitution from public to private consumption, i.e. the private sector does not compensate with higher private consumption for the withdrawal of publicly provided consumption commodities. Any impact on consumption will be the result of indirect changes in the macroeconomic environment that the private sector faces. Without the assumption of additive separability the various effects of government policy would be more difficult to disentangle.

The constancy of the terms of trade implies that the interest rate is the same in both countries. Its evolution is the same as in the single country case, see Figure 6.3. This implies from (5.8) and (5.9) that the evolution of ξ is the same in both countries, because the interest rate is identical and the terms of trade are the constant over time. However, the fact that the evolution of ξ is identical does not imply that $\xi_t = \xi_t^*$ $\forall\, t$. Since the reluctance rate ξ is not a predetermined variable, it will not be the same unless the terminal conditions are identical in the domestic and the foreign economy. From the discussion on page 32 the terminal condition could loosely be interpreted as the long run unity of the marginal propensity to consume out of income. Since domestic income rises faster then foreign income, the terminal condition can not be the same. A natural initial idea would be to argue that since in the domestic economy, income grows faster, consumption must grow faster as well, which would mean that the marginal propensity to consume must increase. However that is not the case here, since the increase in ξ corresponds to an slowdown in consumption, rather than an increase.

The rise of ξ in the fast-growing country may be counter-intuitive, but can be quite easily explained referring to the fundamental equation (4.12). Imagine a dynamic form of this equation as

$$1 + r(t-1) = (1 + \text{д})\,(1 + n(t))^{\sigma} \tag{6.3}$$

and substitute into (3.20) to see that

$$1 + \xi(t) = 1 + \frac{1 + n(t+1)}{1 + r(t)} + \frac{(1 + n(t+1))\,(1 + n(t+2))}{(1 + r(t))\,(1 + r(t+1))} \dots \tag{6.4}$$

clearly suggesting that the higher is growth, the higher is the reluctance to consume out of wealth.

The numerical results suggest that stability of the model requires that in the domestic country $\xi_1 = 230\%$ whereas $\xi_1^* = -250\%$. These appear to appear to be large but are in fact quite small changes to the initial steady-state value of 24.752. The movement of ξ is illustrated in Figure 6.4. The evolution of ξ from period 2 onwards is the same in both countries and is identical to the evolution of ξ in the reference case for the domestic economy.

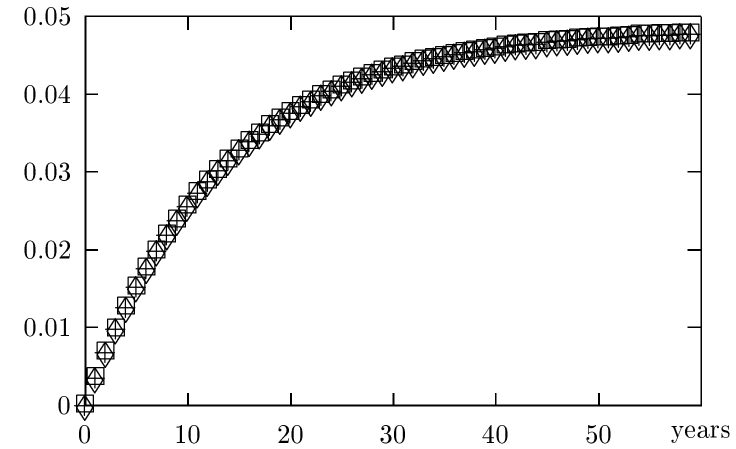

Figure 6.3: $\mathrm{r}_t = 1, \ \forall\, t \geq 1$: $r_{t-1} \leftrightsquigarrow \Diamond$, $r^*_{t-1} \leftrightsquigarrow \Box$, $\tilde{r}_{t-1} \leftrightsquigarrow +$

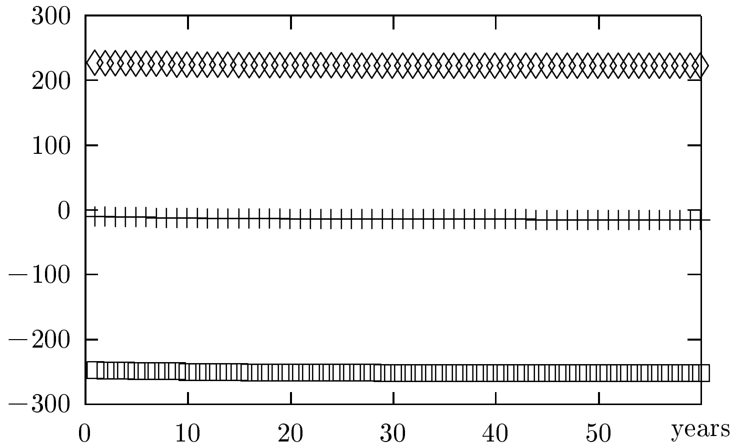

Figure 6.4: $\mathrm{r}_t = 1, \ \forall\, t \geq 1$: $\xi_t \leftrightsquigarrow \Diamond$, $\xi^*_t \leftrightsquigarrow \Box$, $\tilde{\xi}_t \leftrightsquigarrow +$

The most important feature that distinguishes the open economy from the closed economy is that ξ is allowed to raise in the domestic economy and fall in the foreign economy, whereas in the domestic reference case it remains close to the steady-state reference value. An increase (decrease) in the reluctance rate implies that domestic (foreign) residents consume less (more) out of accumulated wealth. Consumption itself is determined by the stock of wealth and the propensity to consume from it. For the stock of wealth—which is not predetermined—I note that through the increase (decrease) in growth, the domestic (foreign) human wealth jumps upwards (downwards). Since the current growth rate is predetermined, the jump in human wealth reaches its peak in $t = 1$ rather than in $t = 0$.

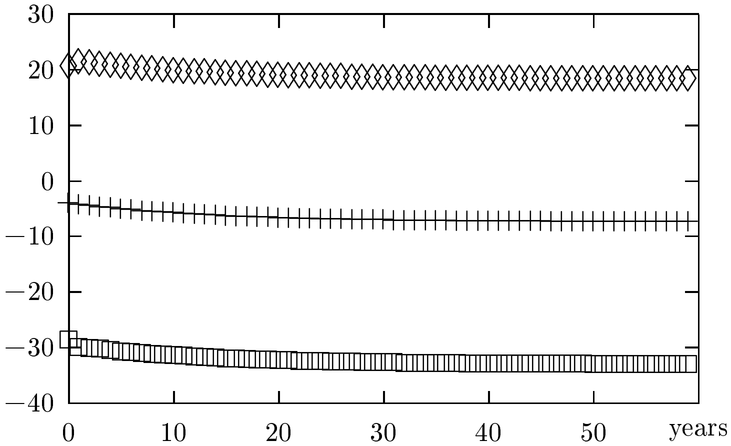

Figure 6.5: human wealth when $r_t = 1,\ \forall\, t \geq 1$: $w_{t-1}^{h} \leftrightsquigarrow \Diamond$, $w_{t-1}^{h*} \leftrightsquigarrow \Box$, $\tilde{w}_{t-1}^{h} \leftrightsquigarrow +$

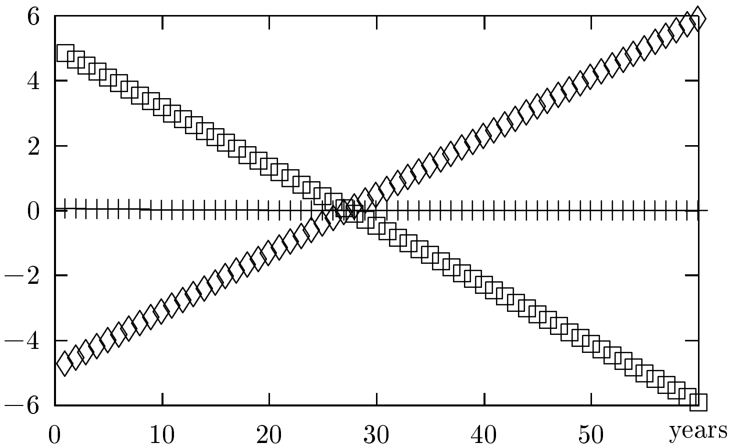

Figure 6.6: consumption when $r_t = 1,\ \forall\, t \geq 1$: $c_t \leftrightsquigarrow \Diamond$, $c_t^* \leftrightsquigarrow \Box$, $\tilde{c}_t \leftrightsquigarrow +$

The initial impact on consumption in the domestic (foreign) economy is the difference of the impact of the increase (decrease) in human wealth and the decrease (increase) of the marginal propensity to consume. As illustrated in Figure 6.6 the latter effect is more important than the former, which implies that consumption falls (rises) initially in the domestic (foreign) economy. This impact effect is the principal cause of the accumulation of assets by one country over the other. The accumulation of assets allows domestic (foreign) consumption to rise (fall). In fact $\lim_{t\to\infty} c_t = \infty$, i.e. consumption is not stationary because asset accumulation is not.

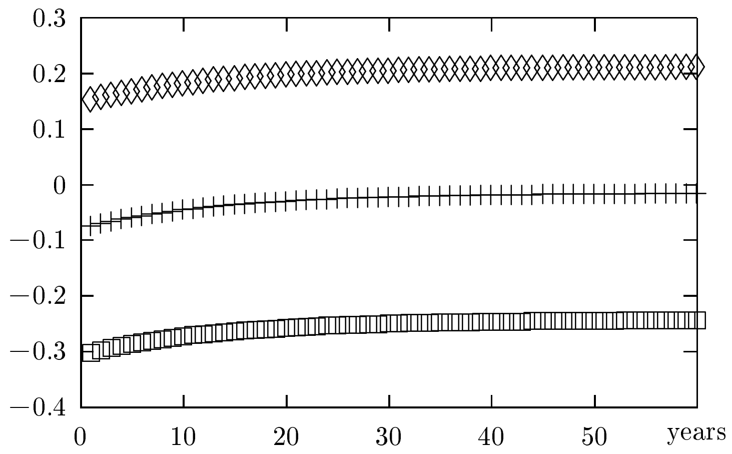

Figure 6.7: investment when $r_t = 1, \ \forall t \geq 1$: $i_t \leftrightsquigarrow \Diamond$, $i^*_t \leftrightsquigarrow \Box$, $\tilde{i}_t \leftrightsquigarrow +$

The discussion has until now left out the supply side of the model. In the state-space representation, investment is determined as a residual. On impact, the decrease in domestic consumption and the increase in foreign consumption mostly cancel out. The difference is the terms of trade effect that is constant both in the long and the short run. The drop in the terms of trade allows for cheaper imports in the domestic economy, and more expensive imports in the foreign economy. As a consequence, the foreign residents reduce their demand for the Home commodity. According to 5.10 this effect increases domestic investment. This is a permanent shock. Hence the drop in the terms of trade permanently fuels a decrease in total consumption demand in the domestic economy, which allows for higher investment, which again allows for higher growth in the domestic economy. This is the key element that explains the divergence of the growth rate and the fact that the impact of the shock is so much larger in the open economy when compared to the closed economy, something that could be considered a puzzle at first. From Figure 6.7 it is also interesting to note that investment increases over time. From Figure 6.8 the increase in investment is by no way sufficient to halt the decline in the capital stock per GDP that is caused by the expansion in GDP, the second term on the right hand side of equation (5.5). In the foreign economy, the decline of the capital stock occurs through the lack of investment, here the slowdown in growth moderates the fall in the per-GDP capital stock.

In the domestic economy, the infrastructure stock increases from period 1 onwards. In the foreign economy, the infrastructure increases as well, from period 2 onwards, since the growth rate has declined; recall that k^g is measured in per-GDP terms. Figure 6.9 sums up the effect of the fiscal policy change on growth. The

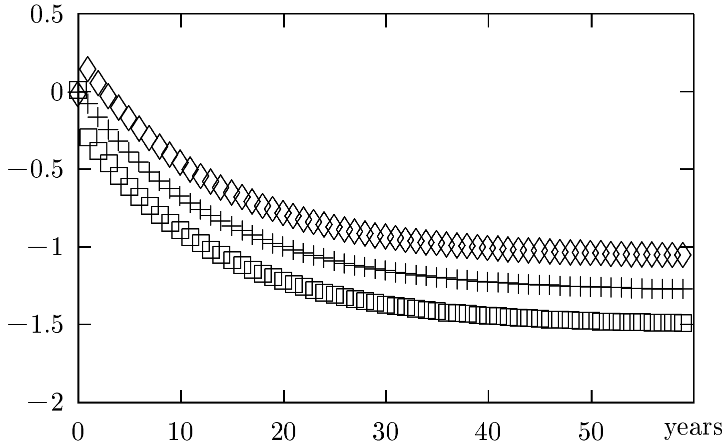

Figure 6.8: capital when $r_t = 1$, $\forall\, t \geq 1$: $k_{t-1} \leftrightsquigarrow \Diamond$, $k^*_{t-1} \leftrightsquigarrow \Box$, $\tilde{k}_{t-1} \leftrightsquigarrow +$

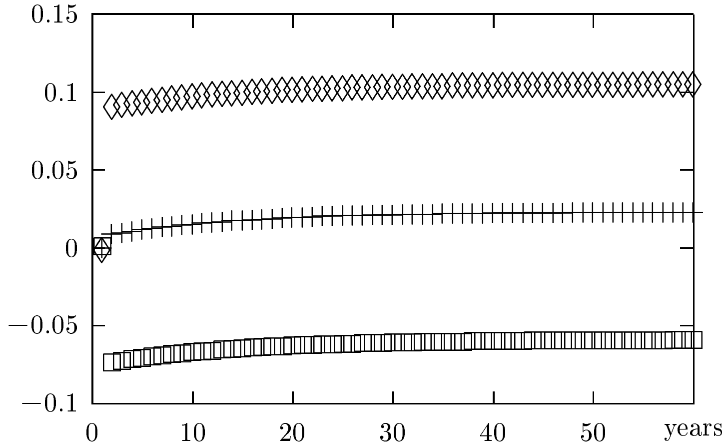

Figure 6.9: growth n_t when $r_t = 1$, $\forall\, t \geq 1$: $n_t \leftrightsquigarrow \Diamond$, $n^*_t \leftrightsquigarrow \Box$, $\tilde{n}_t \leftrightsquigarrow +$

crucial importance of the openess of the economy figures promiently here. Contrary to the situation that would prevail if the economy was closed, in a two-country world the growth effect is much more important. A fall in the terms of trade occurs. It makes the domestic commodity more expensive. In this linearized version of the model, the budget shares do not remain constant. The domestic economy pays its imports cheaper, thus leaving room for additional investment. That additional investment fuels growth. Another part of that reduced expenditure is spent on accumulating assets over the foreign economy. To ensure that high growth in the domestic economy is compatible with slow growth other economy, the stock of foreign assets in the domestic economy, as well as its consumption expediture increases beyond all bounds. This ensures that the domestic economy eventually consumes the additional output that it generates.

6.2 A shock to taxation

The next shock to consider is a once-and-for-all increase in taxation in the home country. This shock has more complicated effects. Since the interest agreed upon in period t is subject to taxation at period $t+1$, there is an additional element of surprise to the private sector if the government changes the tax rate in period 1. To avoid that additional complication, I consider here an increase of the tax from period 2 only. To further simplify, I consider an increase in tax that is used to augment government consumption only, i.e., there is no change in infrastructure expenditure. Thus I avoid the problem that at unchanged r this shock has a component that pushes the economy like the shock discussed in Subsection 6.1.

The issue of source-based vs. residence-based taxation should be kept in mind here. Under the residence principle, the pre-tax interest rates must be equalised internationally if the terms of trade remain constant. Since the pre-tax interest rate is equal to the marginal product of capital that would imply that the marginal product of capital is equalised in all countries and that the international allocation of investment is efficient. I have adopted source-based taxation here. This will allow countries to impact on pre-tax interest rates using domestic taxation, a view that is more realistic. It implies that the post-tax returns are equalised when the terms of trade are constant.

When the tax is imposed the pre-tax interest rate rises, but not by as much as the tax increase, thus the post-tax interest rate r actually declines. In the domestic economy this is the effect of the first term on the left-hand side of (5.3). In the foreign economy, the tax effect is absent. From Figure 6.3, in the absence of a terms of trade effect, the interest rate would fall, then recover to some extent, but not enough to come back to the baseline level. Note that the interest rate falls in period 1, i.e. before the impact of the shock. The differences of interest rates visible in Figure 6.10 are thus explained by the changes in the terms of trade, following the □ed line of Figure 6.2.

The rise in government consumption spending should be expected to make the domestic commodity realitively more expensive. But on impact, we observe a rise in

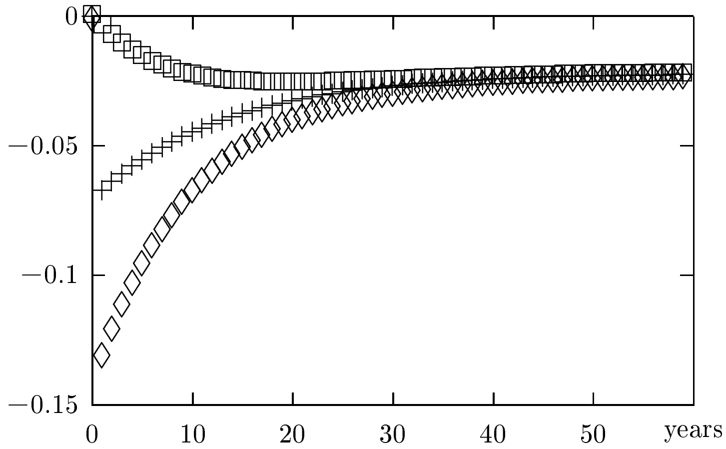

Figure 6.10: interest rate when $g^{c}_{t+1} = \tau_{t+1} = 1,\ \forall\, t \geq 1$: $r_{t-1} \leftrightsquigarrow \Diamond$, $r^{*}_{t-1} \leftrightsquigarrow \Box$, $\tilde{r}_{t-1} \leftrightsquigarrow +$

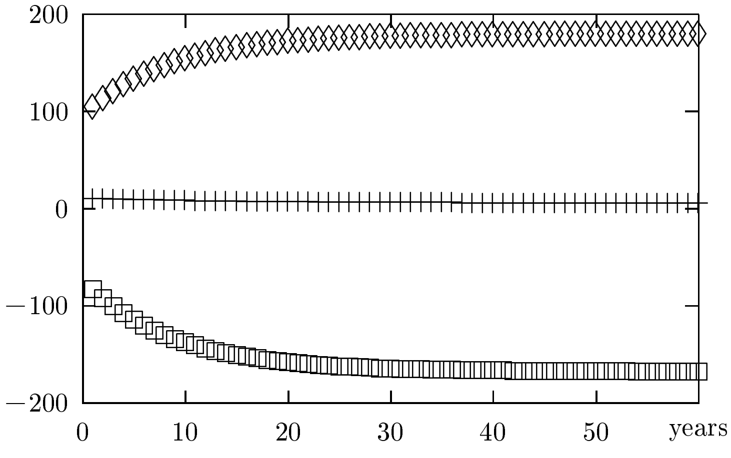

Figure 6.11: reluctance rate when $g^{c}_{t+1} = \tau_{t+1} = 1,\ \forall\, t \geq 1$: $\xi_{t} \leftrightsquigarrow \Diamond$, $\xi^{*}_{t} \leftrightsquigarrow \Box$, $\tilde{\xi}_{t} \leftrightsquigarrow +$

the terms of trade (see Figure 6.2), because of the divergence of interest rates that appears with the imposition of the tax at home. However, in the long run the terms of trade fall and end up lower than the baseline. From Figure 6.10 we see that the domestic interest rate falls by much more than the foreign interest rate. However both interest rates converge after 30 years and the resulting drop in the long run equals the drop experienced in the single country reference case. Figure 6.11 shows that the changes in the marginal propensity to consume are very similar to the ones observed in Figure 6.4. The initial impact in not as large because the initial rise in p pushes for an increase in domestic consumption. However this effect does not dominate. Similarly, the downward movement of the interest rate should increase the reluctance to consume wealth, but this effect overshadowed by the open-economy divergence of ξ.

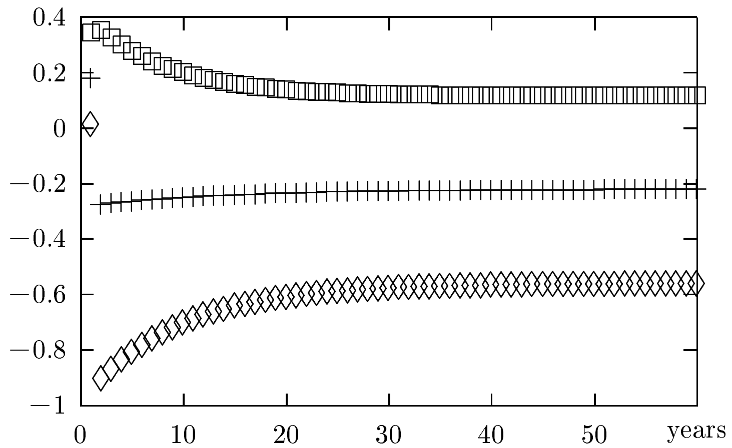

Figure 6.12: investment when $g^c_{t+1} = \tau_{t+1} = 1,\ \forall\, t \geq 1$: $i_t \leftrightsquigarrow \Diamond$, $i^*_t \leftrightsquigarrow \Box$, $\tilde{i}_t \leftrightsquigarrow +$

In the foreign economy, the fall in the post-tax interest rate increases investment. The initial increase in the relative price for the foreign commodity also contributes to the increase in investment. By comparison with the single country case, we can see that the price effect accounts for about 50% of the initial rise. This effect dissappears in later period and consequently the rise in the foreign capital stock levels off. In the domestic economy, the increase in taxes and the adverse terms of trade effect combine to reduce investment. The initial fall in investment roughly equals the increase in taxation. Overall there is a decline in the domestic capital stock and an increase in the capital stock abroad illustrated in Figure 6.12.

Figure 6.13 illustrate the movement of the infrastructure. $k^g_0 = 0$ since the stock is predetermined. k^g_1 is also zero because the the policy change only occurs in period 2. The domestic (foreign) infrastructure to GDP ratio then increases (declines) because GDP declines (increases). This movement of capital to GDP ratios should be kept in mind when considering the evolution of the private capital stock in 6.14. Lower interest rates make for a sustained increase in the capital stock in the foreign economy. In the domestic economy there is a sustained fall of the private capital stock to GDP ratio despite the fall of domestic GDP.

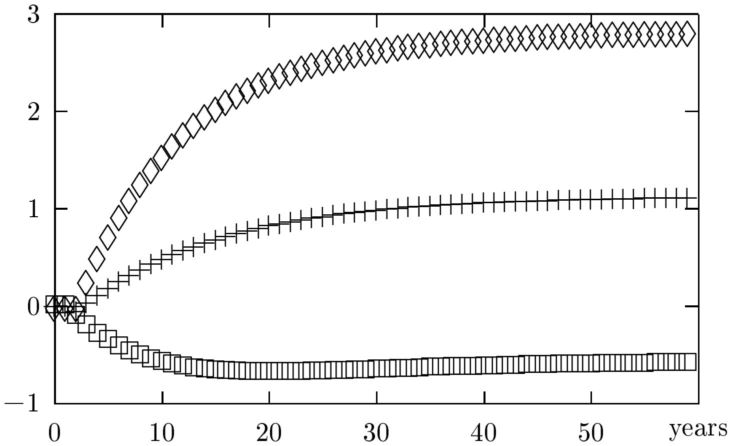

Figure 6.13: infrastructure when $g^{c}_{t+1} = \tau_{t+1} = 1$, $\forall\, t \geq 1$: $k^{g}_{t-1} \leftrightsquigarrow \Diamond$, ${k^{g}}^{*}_{t-1} \leftrightsquigarrow \Box$, $\tilde{k}^{g}{}_{t-1} \leftrightsquigarrow +$

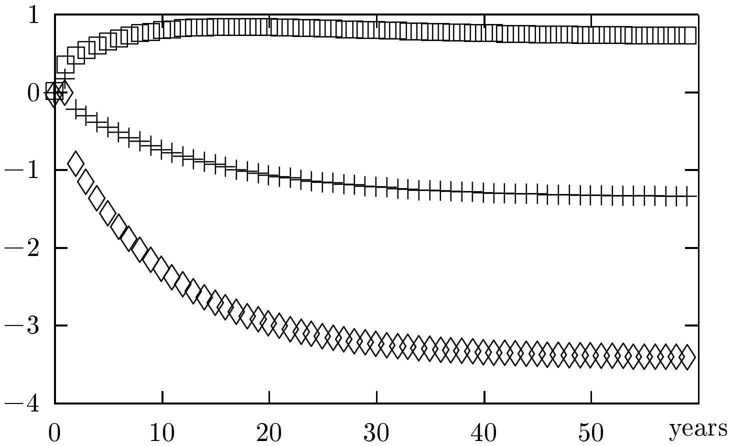

Figure 6.14: capital when $g^{c}_{t+1} = \tau_{t+1} = 1$, $\forall\, t \geq 1$: $k_{t-1} \leftrightsquigarrow \Diamond$, $k^{*}_{t-1} \leftrightsquigarrow \Box$, $\tilde{k}_{t-1} \leftrightsquigarrow +$

The path of the growth rate is drawn in Figure 6.15. n_0 is predetermined. In the next period the increase in taxes is announced, but not implemented. In the single country case private investment increases in that period, because the pre-tax interest rate does not increase fully by the amount of the tax, and the crowding-out effect of increased government consumption on the commodity market will only impact in the next period. In the foreign economy, the impact of lower interest rates and the upward jump in the terms of trade increase investment and therefore growth, but the latter effect fades out. In the domestic economy, higher taxes discourage investment and therefore the growth rate is lower, despite the long run fall in the terms of trade.

The main puzzle of these results are here one country growing slower than its neighbour but acquiring assets over it. The main difference with case of the increase in infrastructure is that the path of consumption does not immediately follow a straight line since the transitional dynamics the terms of trade result in higher domestic consumption and lower foreign consumption. But in the long run the picture is the same as Figure 6.6. As suggested by Figure 6.1 the accumulation process is almost identical in the cases of raising the government infrastructure and of raising government spending. It is puzzling to note that these two shocks have very little in common that would point to the origin of the result that foreign assets accumulate.

It is easy to see why the accumulation of assets in the first period is compatible with the other simulation results. Using (5.10) in (5.7), the symmetry of the calibration and some initial conditions, it can be shown that

$$\frac{\mathcal{A}_1}{c} = \frac{\xi_1^* - \xi_1}{1+\xi} + (1-\alpha)\, p_1$$

which is a positive term. What appears puzzling is that the shock is persistent in the sense that the accumulation of foreign assets remains constant in all period. This is addressed in the next Subsection.

An interesting experiment is to simultaneously allow for the infrastructure shock in the foreign economy and the public consumption shock in the domestic economy. In that case the long run growth rate in the domestic economy declines by almost .2%, whereas the foreign economy's growth rate increases by .13%. The annual increment in the foreign assets is only .264%, which is substantially lower then the long 4.48% for the infrastructure shock and 4.745% for the tax increase. This clearly demonstrates yet again that asset accumulation is not directly related to the growth gap between countries.

Let us summarize what we have learned from these simulations. The response to shocks produces both changes to the terms of trade and to foreign lending. Terms of trade changes level out in the long run. The terms of trade move to a new steady-state value, and since the solution of the model is not affected by the value of the terms of trade, the effect of terms of trade changes is transitory. However the changes to foreign lending are permanent. Foreign lending tends in the long run towards showing a unit root behaviour. Consumption shows the same long run unit root because it depends on wealth which of course includes foreign assets.

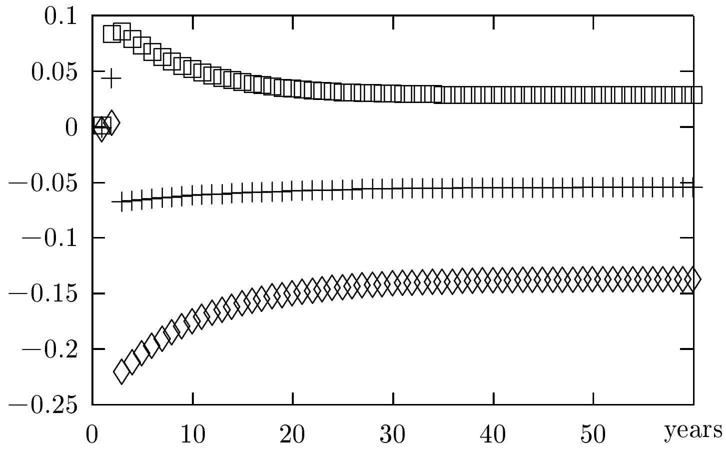

Figure 6.15: growth rate when $g^{c}_{t+1} = \tau_{t+1} = 1,\ \forall\, t \geq 1$: $n_t \leftrightsquigarrow \Diamond$, $n^*_t \leftrightsquigarrow \Box$, $\tilde{n}_t \leftrightsquigarrow +$

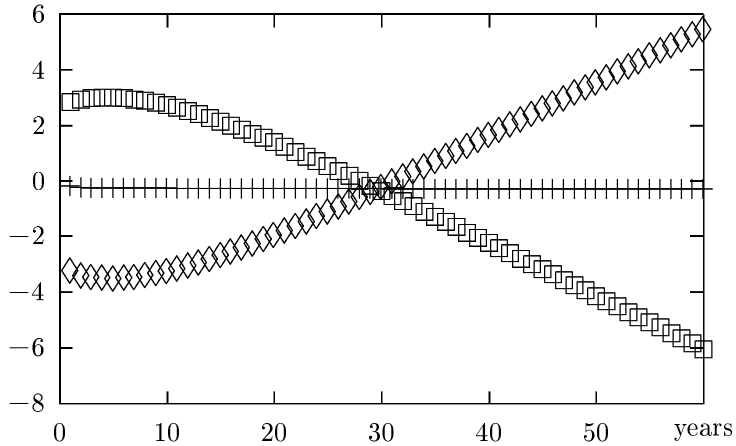

Figure 6.16: consumption when $g^{c}_{t+1} = \tau_{t+1} = 1,\ \forall\, t \geq 1$: $c_t \leftrightsquigarrow \Diamond$, $c^*_t \leftrightsquigarrow \Box$, $\tilde{c}_t \leftrightsquigarrow +$

6.3 Discussion of shock equilibria

The most striking feature of the simulation presented here is the unit root in foreign assets. That is, when we have a permanent shock, foreign assets evolve as

$$л_{\mathsf{t}} = л_{\mathsf{t}-1} + \text{constant} \tag{6.5}$$

It is easy to dismiss this feature as an artifact of linearisation. In this subsection, I will show that the relationship (6.5) is a feature of any long run solution of the model and not related to linearisation.

Let's first return to the subject of Subsection 4.1, the conditions for existence of a steady state. In particular I show a condition (4.12) that links the long run interest rate to the long run growth rate. This equation applies in a steady state of a closed economy. To see if it extends to an open economy, I write the three equations of Table 4.1 that change from the closed to the open economy

$$c = \frac{щ\,(1+r)}{(1+n)\,(1+\xi)} \tag{4.1'}$$

$$щ = щ^{\mathrm{h}} + k + л \tag{4.2'}$$

$$1 - \tau = c + i + \frac{n-r}{1+n}\,л \tag{4.11'}$$

It is then straightforward to see that with these equations (4.13) becomes

$$\begin{aligned} 1-\tau = {} & \left[(1-\tau)\,\kappa\,\frac{1+n}{\delta+r} + (1-\kappa)\,(1-\tau)\,\frac{1+n}{r-n} + л\right]\frac{1+r}{(1+n)\,(1+\xi)} \\ & + \frac{(1-\tau)\,\kappa\,(n+\delta)}{r+\delta} + \frac{n-r}{1+n}\,л \end{aligned} \tag{4.13'}$$

Again, equation (4.17) gives a solution to this equation. Thus if a steady state exists where foreign assets are in the steady state, then (4.12) has to hold, if we exclude the other solution that gives zero consumption when foreign assets are zero. The calibration of the model simulated in this section is of course built on the assumption that (4.12) holds both for the domestic and the foreign economy.

To study the accumulation of foreign assets consider (3.34). Substituting in (3.19) I arrive at

$$\begin{aligned} Л(\mathsf{t}) = {} & \frac{(1+r(\mathsf{t}-1))\,\xi(\mathsf{t})}{1+\xi(\mathsf{t})}\,Л(\mathsf{t}-1) + (1-\tau(\mathsf{t}))\,Q(\mathsf{t}) - I(\mathsf{t}) \\ & - \frac{(K(\mathsf{t}-1) + Щ^{\mathrm{h}}(\mathsf{t}-1))\,(1+r(\mathsf{t}-1))}{1+\xi(\mathsf{t})} \end{aligned}$$

Write this equation is per-GDP form

$$\begin{aligned} л(\mathsf{t}) = {} & \frac{(1+r(\mathsf{t}-1))\,\xi(\mathsf{t})}{(1+\xi(\mathsf{t}))\,(1+n(\mathsf{t}))}\,л(\mathsf{t}-1) + (1-\tau(\mathsf{t})) - i(\mathsf{t}) \\ & - \frac{(k(\mathsf{t}-1) + щ^{\mathrm{h}}(\mathsf{t}-1))\,(1+r(\mathsf{t}-1))}{(1+n(\mathsf{t}))\,(1+\xi(\mathsf{t}))} \end{aligned}$$

Now assume that r, n and ξ are in the steady state, then from Proposition 4.1 we know that for the parameter choice that corresponds to a closed economy, we have

$$a(\mathrm{t}) = a(\mathrm{t}-1) + 1 - \tau(\mathrm{t}) - i(\mathrm{t}) - \frac{k(\mathrm{t}-1) + w^{\mathrm{h}}(\mathrm{t}-1)}{\xi} \tag{6.6}$$

That is where the unit root appears! In any long run steady state where taxes and investment are constant shares of GDP, where growth and interest rates are constant and therefore when capital and human wealth are constant multiples of GDP, foreign assets will evolve as a unit root.

In the starting lines of this subsection, I have shown that for steady state with positive consumption, equation 4.12 still holds when foreign assets are a constant fraction of GDP. Then I show that this implies that foreign assets are following a unit root. Clearly if the foreign assets follow a unit root, they can not be constant. This circular reasoning shows that in fact there is no *stable* steady state where both all aggregates and interest and growth rates are constant *and* where foreign assets are stable. Anything else can be stable, but then foreign assets won't. That is exactly what my Figure 6.1 shows. This striking phenomenon is rooted "deep" in the model property. It has nothing to do with either a particular choice of parameters or with the linearisation.

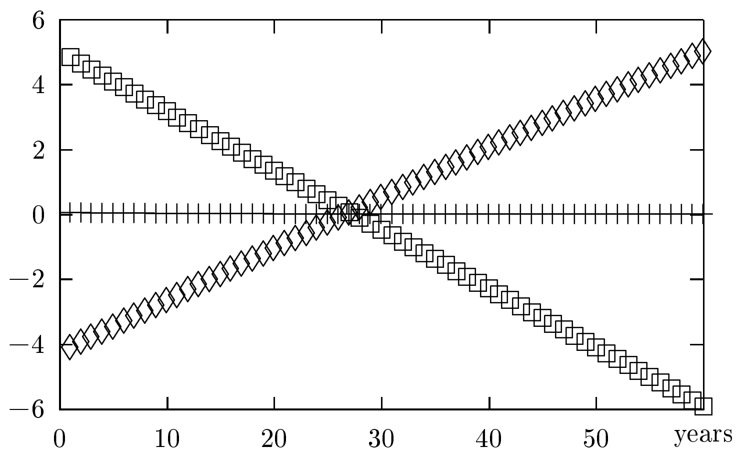

Figure 6.17: consumption when $\mathrm{r_t} = 1,\ \forall\,\mathrm{t} \geq 1$: $c_\mathrm{t} \leftrightsquigarrow \Diamond$, $c^*_\mathrm{t} \leftrightsquigarrow \Box$, $\tilde{c}_\mathrm{t} \leftrightsquigarrow +$, with risk premium

The unit root is difficult to admit for the country that is loosing assets, because there shall come a time when total assets decrease below zero, and since consumption is a positive multiply of total assets, it becomes negative. It is quite easy to avoid that feature by introducing a risk premium. Replace (5.3) by

$$\begin{aligned} r_\mathrm{t} = & -\bar{\epsilon}\,\gamma\,k^{\gamma-1}\,k^{\mathrm{g}\,1-\gamma}\tau_{\mathrm{t}+1} \\ & -\bar{\epsilon}\,(1-\tau)\,\gamma\,(1-\gamma)\,k^{\gamma-2}\,k^{\mathrm{g}\,1-\gamma}\,k_\mathrm{t} \\ & +\bar{\epsilon}\,(1-\tau)\,\gamma\,(1-\gamma)\,k^{\gamma-1}\,k^{\mathrm{g}-\gamma}\,k^{\mathrm{g}}_\mathrm{t} - \psi\,a_{\mathrm{t}-1} \end{aligned} \tag{5.3'}$$

i.e. allow for the accumulation of foreign assets to reduce the domestic interest. For the foreign country I do the opposite, i.e., *add* a term $\psi \mathcal{A}_{t-1}$. Simulations show that a ψ as small as 10^{-8} is sufficient to stabilise foreign assets. To ensure that consumption in the slow growing country remains positive, I need a much larger figure. For the case of the infrastructure shock of Subsection 6.2, setting $\psi = .00003$ ensures that consumption remains positive. Basically such a value suggests that a country that has a foreign debt that 10 times its annual GDP would require a risk premium of .03%. Of course this is a value that would appear on the low side, but all I wish to illustrate here is that I can break the unit root with such a low value to a sufficient degree that the resulting consumption in the slow growing country remains positive. The corresponding graph for the first 60 periods is given in Figure 6.17. It is almost the same graph as Figure 6.6, but now the lines for c_t and c_t^* are strictly concave. In the limit, the domestic consumption rises by 57.5092%, whereas the foreign consumption declines by 57.4792%. The feedback of the interest rate on foreign assets ensures differential growth without one country's consumption becoming negative.

7 The optimal and time-consistent policies for the single economy

At period 0, the economy is in the steady state of Table 4.1 with the values of Table 5.2 found on page 44. These are the long-run values used for the linear-quadratic approximation of the model. From period 1 onwards, a maximising government intervenes. There are several ways in which this assumption can be justified. One idea would be that the original steady-state corresponds to an optimum reached when preferences were different, i.e. a shift in preferences occured at period 1. If that original steady state would be the outcome of a time-consistent optimisation process, another approach to justify to study the optimisation exercise would be to imagine that a precommitment mechanism has been found.

Whatever assumption one uses to justify the meximisation, there are two regimes, depending on whether or not we allow the government to precommit. If the government can precommit, i.e. set all values of future instruments in the current period, it will maximise

$$U_1 = \sum_{\mathfrak{t}'=1}^{\infty} \mathring{\varrho}^{\mathfrak{t}'-1} u_{\mathfrak{t}} \tag{7.1}$$

where $u_{\mathfrak{t}}$ is given in equation (5.19). When the government cannot precommit itself to a future policy, it must act each period to maximize its welfare function, given that a similar optimization problem will be carried out in the next period. Formally, the government maximizes at time $\mathfrak{t} \geq 1$ a welfare function $U_{\mathfrak{t}}$ such that

$$U_{\mathfrak{t}} = u_{\mathfrak{t}} + \mathring{\varrho}\, U_{\mathfrak{t}+1} \tag{7.2}$$

where $U_{\mathfrak{t}+1}$ is evaluated on the assumption that an identical optimization exercise is carried out from time $\mathfrak{t}+1$ onwards. The solution to this problem is found by dynamic programming and, unlike the precommitment policy, leads to a time consistent trajectory or rule for instruments. I label this solution "ST", and label the precommitment solution "SP".

In this section I will present a series of graphs that illustrate the results. One important presentational issue arises. Since all variables are expressed in per GDP terms, saying that in one regime per GDP consumption is smaller than in another does not imply that the actual gross flows of consumption are smaller. It is possible that one regime will have a smaller consumption to GDP ratio but allows for a larger consumption flow because for a larger growth rate. For a meaningful comparison of regimes, it is therefore important to introduce figures have been "grossed up" to reflect the evolution of aggregates, rather than the per-GDP values. Pick any aggregate X in the model, and normalize $Y(1) = 1$. One way to perform the calculation of the grossed up variable would be to calculate

$$X_{\mathfrak{t}} = x_{\mathfrak{t}} \prod_{\mathfrak{t}'=1}^{\mathfrak{t}} (1 + n + n_{\mathfrak{t}}) \tag{7.3}$$

This would give the evolution of the real aggregate, when the linear-quadratic policy calculation leads to the sequence of GDP growth rates $(n_1, n_2, \ldots, n_t)$ and the series of the per-GDP aggregates is $(x_1, x_2, \ldots, x_t)$. However that is not what the programme actually computes because the programme calculates a linear quadratic approximation of the real aggregate. The "real" aggregate is

$$X(t) = x(t)\left[\prod_{t'=1}^{t-1}(1 + n(t'+1))\right] \quad t' > 1$$

This approximation is

$$X_t = x_t\,(1+n)^{t-1} + x\,n_t^b\,(1+n)^{t-2} \tag{7.4}$$

In the following I have used this approximation. In each period, this number reports a linear approximation of the ratio of the aggregate as a percentage of income in the original steady-state i.e. the state that would prevail of the government would not have changed its policy on any date. It should be noted that considering (7.4) rather than (7.3) implicitly introduces a bias towards a slow growing but high consumption/GDP ratio regime, because the former takes no account of the non-linear nature of economic growth. This bias is increased by the incomplete root of n^b required to circumvent the accumulator problem.

The calculations seek to maximise (5.19), with a startup penalty of 10^{-9} for the change of taxes in the initial period. That penalty is required for the solution of the optimal control problem.

Table 7.1 gives long run values for both regimes. In the SP regime, taxation increases, whereas in the ST regime taxation falls. The long run deviations in the tax rate from the initial state are rather small in both regimes. The impact of optimization on the investment share is much more substantial. In the long run, the SP regime neglects infrastructure investment, the utility in the long run is propped up by government consumption. The SP government will have invested into infrastructure quite heavily in the earlier periods, such that as a fraction of GDP the infrastructure stock increases, but the increase is small. The time-consistent policy increases investment in infrastructure in all periods.

Since taxes have fallen the ST regime allows for greater private consumption and private investment in the long run. Under SP the share of consumption and investment are reduced. However since the SP regime leads to higher investment in the earlier periods, the fall in the capital stock is not as large as in the ST regime.

The interest rate remains almost constant under SP, but rises by almost half a percentage point under ST. Despite that increase in the interest rate, the fall in the desired capital stock is limited because there is a tax cut. Under ST there is a fall in human wealth through the increase in the interest rate, but the reluctance rate ξ falls as well such that the overall impact of the time-consistent policy is an increase in consumption. Finally the most important effect is on the growth rate. Whereas under the optimal regime the growth rate falls from 2% to 1.86%, it *increases* under the time-consistent regime under to almost 2.5%. Of course this has

	SP	ST
τ_∞	1.110	−2.913
г_∞	−2.272	8.475
r_∞	−0.093	0.486
n_∞	−0.140	0.484
k_∞	−1.828	−1.666
k^{g}_∞	1.255	3.648
$\text{ш}^{\mathrm{h}}_\infty$	1.691	−40.823
ξ_∞	27.811	−145.851
i_∞	−0.473	1.165
g^{i}_∞	−0.100	0.816
c_∞	−0.637	1.748
g^{c}_∞	1.211	−3.729
ш_∞	−0.137	−42.490
I_∞	−2.757	9.062
G^{i}_∞	−1.187	4.573
C_∞	−9.057	30.852
G^{c}_∞	−0.722	2.950
K_∞	−40.663	132.577
K^{g}_∞	−17.221	67.516
Ш_∞	−208.536	677.898
$\text{Ш}^{\mathrm{h}}_\infty$	−167.873	545.321
U_∞	−3429.605	−15225.871
U_1	−6995.045	−39419.355

Table 7.1: The long-run percentage deviations

welfare implications that dwarf the changes in the individual components of GDP. In the longer run, the gross (as opposed to per-GDP) flows of private and public consumption will be much higher under the ST regime than under SP. The long run equilibrium of SP displays slow growth and excessive public consumption of the kind one could relate to the former communist Eastern Block countries.

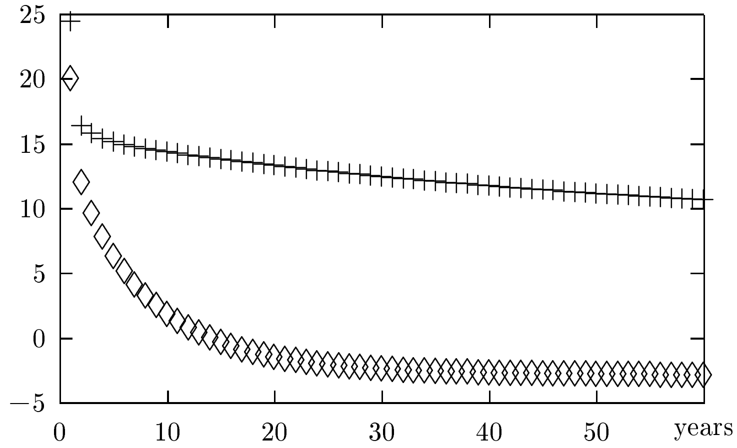

Figure 7.1: r_t, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

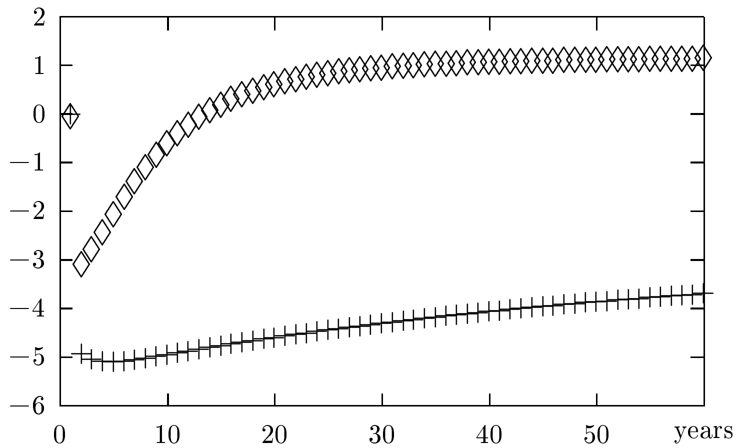

Figure 7.2: taxes τ_t, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

Figure 7.1 shows that both regimes increase the investment share of government spending initially but only ST keeps the investment effort going, in SP the share of investment in infrastructure in government spending drops below the benchmark after 10 years and remains there.

As evident from Figure 7.2, both regimes initially cut taxes to crowd in private investment, but only the ST regime makes the tax cut a permanent one. Both regimes also substantially increase government investment spending, but the SP regime reverses the investment share to levels below the baseline after about 15 years.

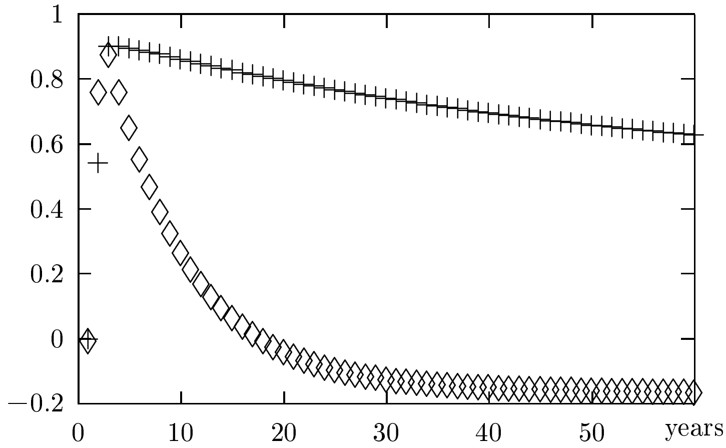

Figure 7.3: growth n_t, SP ⇜⇝ ◊, ST ⇜⇝ +,

0

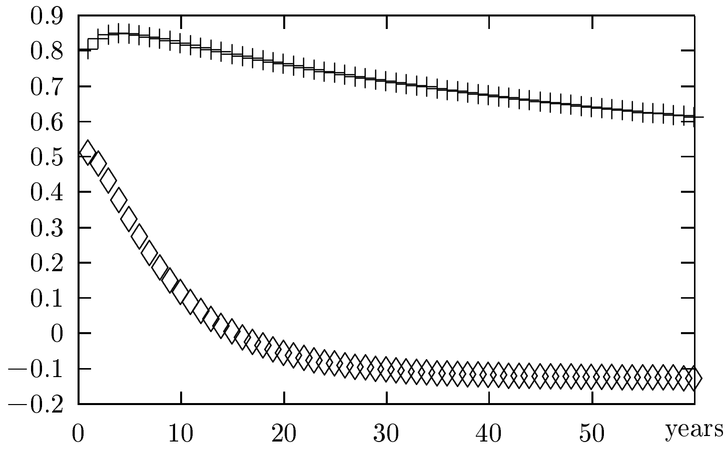

Figure 7.4: interest rate r_t, SP ⇜⇝ ◊, ST ⇜⇝ +,

Figures 7.3 and 7.4 show effects on growth rates and interest rates. Note that $n_1 = 0$ for all regimes since the growth rate is initially predetermined. The optimal regime raises the growth rate initially quite substantially, but after about 20 years all periods that follow are so heavily discounted that growth is no longer worth much sacrifice in the preceding periods, thus growth declines to levels that are below that of the initial steady state. Under SP the movement of the interest rate closely follows the movement of the growth rate. Under the ST regime the rate of growth increases by almost 1% in the initial periods and is slow to come down to its long run change of .484%. In both regimes, the evolution of the interest rate anticipates the movement of the growth rate, because the installed capital stocks at the end of the period determine the current rates of interest, but the growth rate of the next period.

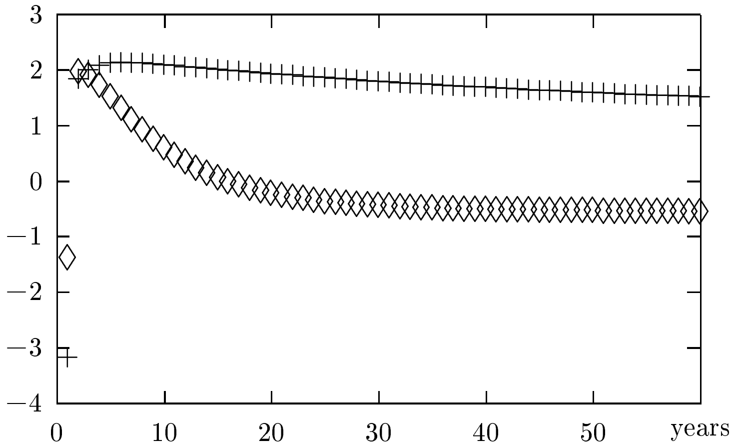

Figure 7.5: Investment i_t, SP ⇜⇝ ◊, ST ⇜⇝ +,

Figure 7.5 shows the trajectory of investment in per-GPD terms. In the first period investment falls because of the immediate impact of the interest rate and the delay in the respone of the infrastructure stock. After about 15 years, the policy of stimulating investment is reversed in the SP regime, investment falls below the baseline because of higher taxes and lower infrastructure investment. The long-run tax cut in the ST regime and the higher stock of infrastructure allows for investment to remain above the baseline.

Figure 7.6 shows private consumption. The comparison of the two regimes is similar to the case of investment. The fall of consumption below the baseline in SP appears already after about eight years, because consumption is forward-looking. Figure 7.7 illustrates that the rise of the reluctance rate over and above the baseline occurs in period 5, even before the consumption is reversed under SP. The long-run increase in taxes dominate the short-run effect of higher private wealth from period 5 onwards. In the ST regime the reluctance rate falls permanently. Despite a fall in private wealth, according to Figure 7.6 the overall result is a rise in consumption because there is a rise in consumption.

The long-run benefits of continued investment in infrastructure, combined with

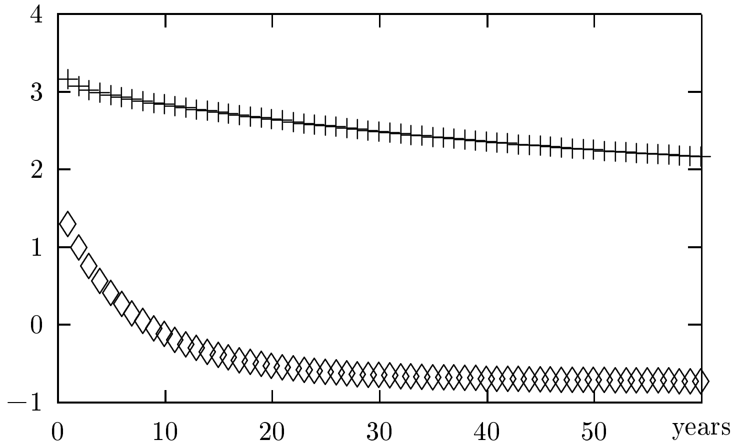

Figure 7.6: Consumption c_t, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

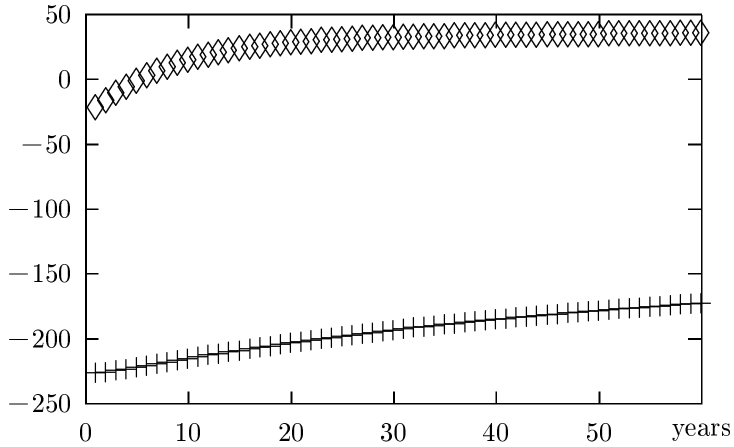

Figure 7.7: reluctance rate ξ_t, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

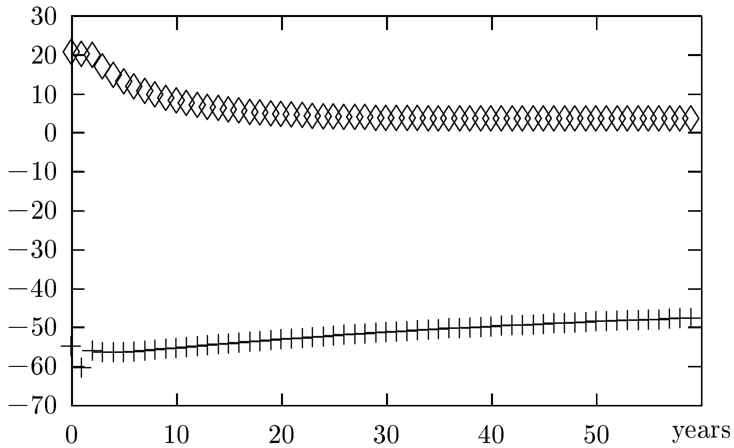

Figure 7.8: wealth w_t, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

a cut in taxes, comes at the price of reduced government consumption. Figure 7.9 shows the drop, but later full recovery of public consumption under SP and the partial recovery under ST. It is instructive to look at a diagram of aggregate government consumption using the approximation (7.4). This is displayed in Figure 7.10. It suggests that whereas the aggregate public consumption remains higher in the SP regime even in the long run—there is no free lunch in the time-consistent regime—the difference is only small in the long run, when one considers that growth is much larger in the ST regime.

To sum up the trajectories of policies, we can say that both policies are very similar at the very beginning. Most aggregates for ST show a substantial change in the initial period that is later very partially reversed. Although in the case of the SP policy the initial change of an aggregate is similar to the ST regime, a shift of policy occurs after about 10 periods and a reversal of the sign of most changes occurs around that time.

The last two rows give figures for the welfare losses. Here U_∞ is the welfare loss on the steady state and U_1 is the cost-to-go, i.e. the welfare loss from the first period to the infinite future. It is worth noticing that the steady-state welfare loss as computed by the progamme is larger under ST than under SC. I am not sure as to why that is the case. Clearly the ST regime appears to be better in the long run. This can be shown when we use the real value of utilty, as opposed to the steady-state approximation that is U_∞. Consider the utility function as

$$U(\mathrm{t}) = \sum_{\mathrm{t}'=0}^{\infty} \varrho^{\mathrm{t}'} \left[\frac{C(\mathrm{t})^{1-\sigma}}{1-\sigma} + \eta \, \frac{G^{\mathrm{c}}(\mathrm{t})^{1-\sigma}}{1-\sigma} \right] \tag{7.5}$$

when in the steady state we have

$$C(\mathrm{t}) = c\,(1+n)^{\mathrm{t}} \quad \text{and} \quad G^{\mathrm{c}}(\mathrm{t}) = g^{\mathrm{c}}\,(1+n)^{\mathrm{t}} \tag{7.6}$$

Provided that $\varrho\,(1+n)^{1-\sigma} < 1$[20] utility in the steady state is given by

$$U = \frac{1}{1-\varrho\,(1+n)^{1-\sigma}} \, \frac{c^{1-\sigma} + \eta\, g^{\mathrm{c}1-\sigma}}{1-\sigma} \tag{7.7}$$

One can the find the growth equivalent of a change from the baseline of Table 4.1 to the long run of ST as .49%, whereas a change to the steady state from the baseline to the the SP steady state is equivalent to an exogenous fall of around .14% in the growth rate. The change in welfare in the steady state is thus important, although admittingly not spectacular.

Another exercise that can be performed on the computer is the calculation of the optimal steady state policy. That is the solution to the optimisation problem faced by a hypothetical policymaker that could choose between different steady states. Numerical simulations show that such a policymaker would choose $\tau = 33\%$ and $\mathrm{r} = 50\%$. Thus the taxation in both regimes is still way off the value that would maximise steady-state welfare.

[20] Barro (1990) and Devereux and Mansoorian (1992) have the same condition. It is of course verified in my calibration.

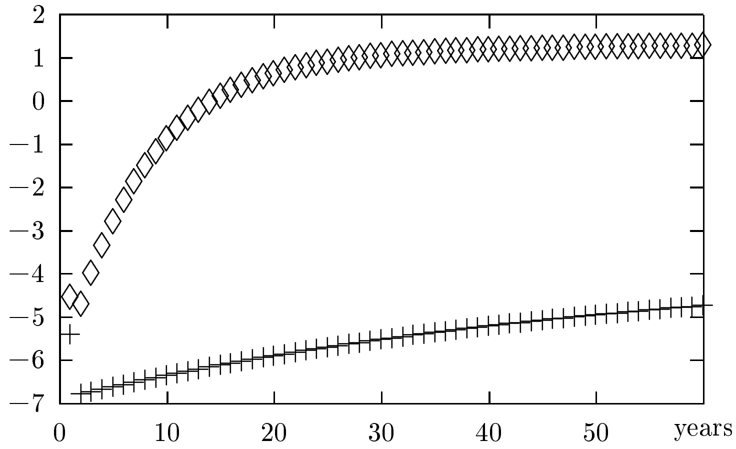

Figure 7.9: public consumption g^{c}_{t}, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

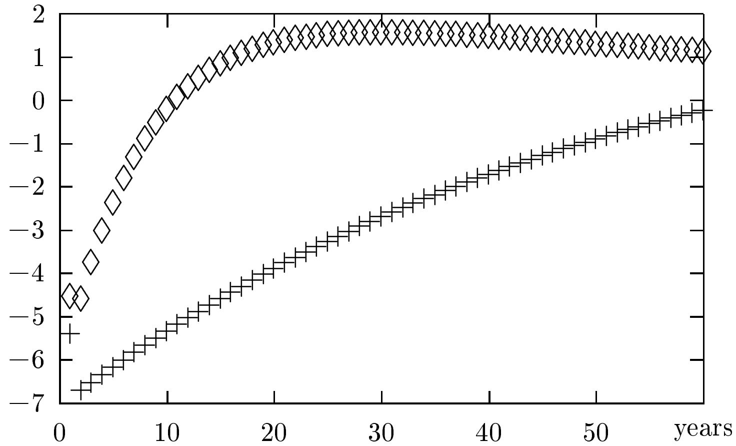

Figure 7.10: public consumption G^{c}_{t}, SP $\leftrightsquigarrow \Diamond$, ST $\leftrightsquigarrow +$,

How sensitive are the results to the calibration? To answer that question I have collected evidence on the long-run values of the model under alternative assumptions about some of the underlying parameters in Table 7.2. In the first two columns there are figures for regimes SP and ST when the elasticity of substitutions has increased up to close to one. The results appear qualitatively the same as the results under $1/\sigma = .5$, but the absolute values of the changes are much larger. The only exception is ξ_∞, but this term is directly affected by the change in σ.

One suspected cause for the suboptimality of the SP regime is the discounting of the periods that are far away in the future. In columns 3 and 4 of the Table 7.2 I examine the case where $д = 0$. Indeed the results of the simulations in the optimal regime show much smaller deviations from the steady-state. The decline in г is much less dramatic and therefore the long run growth rate shows almost no decline. Clearly the SP government that does not discount is more concerned about the long term. But so is the ST government, and as I show in Table 7.2, the strategy that is familiar from the central calibration is intensified. The tax cut is deeper and

	$\sigma = 1.01$		$\text{д} = 0$		$\gamma = .9208$	
	SP	ST	SP	ST	SP	ST
τ_∞	2.70	−12.06	0.26	−9.64	−6.83	−5.54
г_∞	−9.65	30.96	−0.11	23.17	8.53	2.27
r_∞	−0.44	1.28	0.00	1.12	0.69	0.45
n_∞	−0.55	1.80	−0.01	0.99	0.92	0.70
ξ_∞	10.50	−30.48	−4.77	−1368.60	−206.91	−136.30
I_∞	−573.69	1869.88	−0.20	13.50	10.51	8.14
G^{i}_∞	−217.96	708.66	0.03	5.54	3.065	1.25
C_∞	−1564.28	5097.05	−0.38	34.72	32.13	24.48
G^{c}_∞	−381.60	1240.92	0.11	−4.31	0.22	0.844
K_∞	−9721.95	31687.53	−2.86	174.15	137.98	107.61
K^{g}_∞	−3693.72	12009.12	0.639	73.32	32.54	6.66
$\text{Ш}^{\text{h}}_\infty$	−67707.15	220614.61	−19.35	793.22	538.94	421.88
Ш_∞	−77429.10	252302.14	−22.20	967.37	676.92	529.49

Table 7.2: Sensitivity of model

the infrastructure expenditure is more important, such that the capital stocks more rapidly and bring more benefits in the later periods. Therefore the basic qualitative features of the comparison between the regimes remains the same.

In the last column I examine an alternative assumption for the parameter γ. I use the value that would be the optimum in the original Barro (1990) model. In his model, the optimum is time consistent since by assumption taxes do not change over time. In addition, there is no transitional dynamics since the infrastructure stock is assumed to completely depreciate. He then computes the optimal tax rate as $\tau = (1 - \gamma)/\text{г}$. Since this value is achieved with full depreciation, we expect the optimal level of infrastructure to be lower.[21] Thus here we are in a situation where the initial steady state overinvests in public infrastructure. In this case the SP and ST regimes ressemble each other, they both involve a tax cut but an increase in the fraction spent on infrastructure in the long run. In the short run the SP government can use the existing overaccumulated capital stock to satisfy the need for current rather than future consumption implied by discounting. Both government consumption and investment fall as a ratio of GDP in both regimes.

I would qualify this latter case, where government capital stock is too large to begin with, as an irregular one. The absence of any criterion to select between historic steady states from which to start the model is an obvious problem for any dynamic model. If the starting point of the model is in the very early days when government first is created, it would be natural to assume that the infrastructure stock is too small rather than too large. If we pick the starting point along the trajectory of an SP government we know that overaccumulation of infrastructure is not relevant since this regime underaccumulates infrastructure in the long run. If we

[21] See Futagami, Morita, and Shibata (1993) for a discussion of this aspect.

pick the starting point along the trajectory of an ST government, the reoptimisation will not change the trajectory of the government's policy. That follows from the definition of the time-consistent equilibrium. But it goes without saying that if the policy before the first period is irrational then clearly anything could have happened, including overaccumulation of government capital.

The conclusion that the time-consistent regime may fare much better in the longer run should be contrasted with the received wisdom regarding the time-consistency issue. For example Zee (1994), writes (page 132)

> The present paper takes the position that the time-inconsistency problem is worth preventing, because on a practical level its presence would cast doubt on the desirability of implementing governments announced policies and on a conceptual level it would rob much of the substantive content in most dynamic optimization exercises that are routinely carried out in many areas of economic research.

I show that in the absence of debt the time-consistent solution has desirable properties in the long run, most importantly that it generates higher growth for a wide range of parameters. Thus the issue of time inconsistency is important in models of endogenous growth.

8 Cooperation versus non-cooperation

In Section 6 I used a linear approximation of the model to show that without a reaction to policy in the foreign country, a change of policy in the domestic country may leave the foreign country to accumulate foreign debt, up to the point where consumption becomes negative and the utility derived from it becomes undefined. Clearly any meaningful equilibrium in this model requires symmetry in policy. In a model with identical countries, the outcome is inevitably symmetric. From the analytical literature I described in Subsection 2.5 on page 21, it should be expected that the spillover effects for symmetric calibrations are much smaller. In fact that is confirmed by the results in Table 8.1. It lists long run deviations from the steady state for three regimes. TP is the two-country cooperative pre-commitment regime. This regime produce results that are very close to the ones for the SP regime of Section 7.[22] TC is the cooperative non-reputational regime. TN is the regime with-

	TP	TC	TN
r_∞	−0.095	0.486	0.462
n_∞	−0.142	0.484	0.440
k_∞	−1.821	−1.666	−2.483
k^{g}_∞	1.239	−1.666	−2.479
w^{h}_∞	1.809	−40.823	−41.407
ξ_∞	27.860	−145.851	−138.773
r_∞	−2.308	8.475	7.784
τ_∞	1.123	−2.913	−2.538
i_∞	0.478	1.165	1.003
g^{i}_∞	−0.104	0.816	0.799
c_∞	−0.633	1.748	1.551
g^{c}_∞	1.226	−3.729	−3.337
w_∞	0.099	−42.490	−43.500
U_1	−6971.925	−39419.355	−31915.615

Table 8.1: Long run values of cooperative and Nash regimes

out cooperation and without reputation. Details of solution procedures are found in Appendix B. Note that there are no values for the regime with reputation but without cooperation. This is simply because the ACES software does not compute that case. Table 8.1 contains the long run results for the regimes. The cost-to-go U_1 of the cooperative regime is higher than the cost-to-go of the non-cooperative regime. Thus we appear to be in a situation where coordination does not pay. However on inspection of the values the cooperative regime seems to fare better in all

[22] In fact there should be no difference in the two regimes. However the ACES software insists on diverging into asymmetric solutions cooperative reputational regime. I suspect that there is a bug in the software for this particular regime. The workaround that problem was to set $\mathscr{n}_{\mathrm{t}} = 0, \forall \mathrm{t}$. Still with this constraints, some price changes still occur that make for a small divergence. The results presented here are the ones for the domestic country.

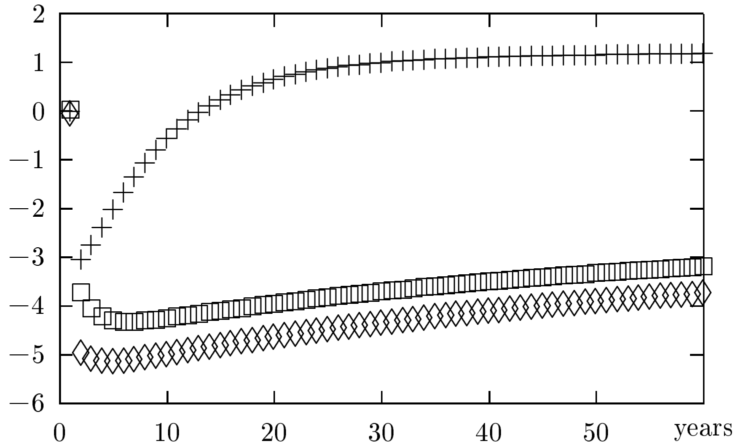

Figure 8.1: τ_t, TC ⇜⇝ ◊, TN ⇜⇝ □, TP ⇜⇝ +,

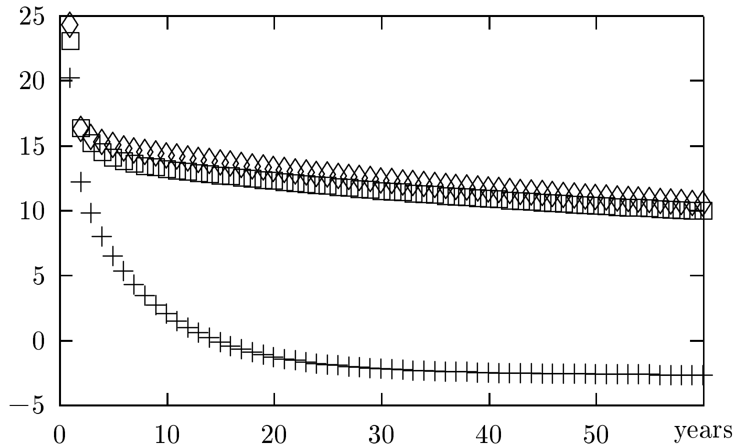

Figure 8.2: r_t, TC ⇜⇝ ◊, TN ⇜⇝ □, TP ⇜⇝ +,

areas but government consumption. Using (7.7), I find that the welfare improvement of the steady state of TN from the baseline is a growth equivalent of .442%. This is smaller than the .486% growth equivalent that a transition to the steady state of TC is equivalent to. Thus in the long run, cooperation is welfare improving, but not along the trajectory. Cooperation improves long-run welfare through lower taxation, higher public infrastructure, a larger capital stock and higher growth. The only component of welfare where the long run of TC is weaker is public consumption. In this respect the comparison between the long run of TC and TN is quite similar to the comparison between between ST and SP. Both ST and TC depress public consumption and fare better in the longer run at the expense of lower felicity in the first periods.

Figure 8.1 illustrates the changes in the tax rate. For technical reasons that are linked to the state-space representation, the initial tax rate is predetermined, i.e. the tax rates is only allowed to vary from period 2 onwards. All regimes initially lower taxes, but the time-consistent regimes lower taxes by more and the tax cut is persistent. Taxes rates reach a minimum in period 5 for TC and period 6 for TN after which they slowly rise, but remain over 2% lower than the baseline. The non-cooperative regime decreases taxes by a smaller amount than the cooperative regime. The fiscal policies of both regimes converge over time, but a small gap remains even in the long run. The TP regime contrasts sharply, the fall in taxation is reversed as early as the second period, and by about 10 periods, the initial cut in taxation becomes a tax hike.

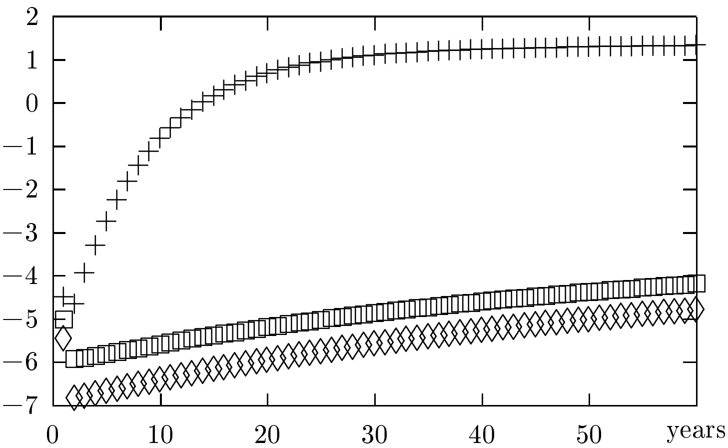

Figure 8.3: g_t^c, TC ⇜⇝ ◇, TN ⇜⇝ □, TP ⇜⇝ +,

Figure 8.2 illustrates that the cooperative regime overinvests more heavily in public infrastructure than the Nash regime, but not by much if we take the distance with the TP regime as a benchmark. The difference between the two time-consistent regimes tends to decline, after 60 periods it is only 2/3 of what it is in the first period, but the difference does not vanish. All regimes show an initial increase in infrastructure expenditure. The time-consistent regimes keep infrastructure spending up in the long run, but the optimal regime TP looses interest in public infrastructure. After about 10 periods the infrastructure spending proportion drops below the baseline and remains there.

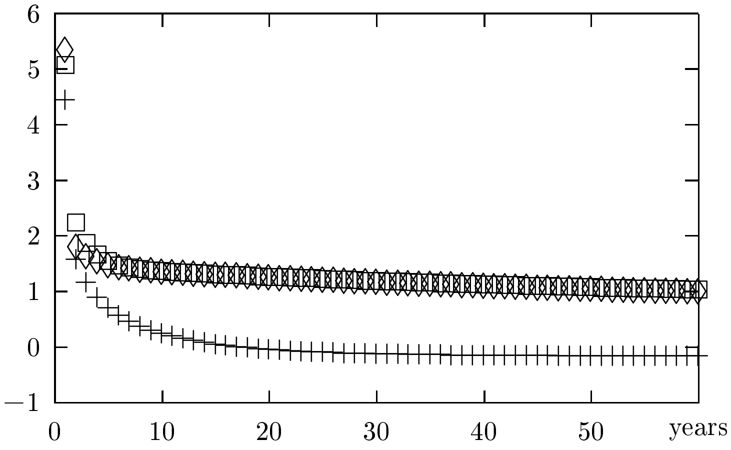

Figure 8.4: g^i_t, TC ⇜⇝ ◊, TN ⇜⇝ □, TP ⇜⇝ +,

The consequences of these policies for the government's consumption and investment are pictured in 8.3 and 8.4. The time-consistent regimes initally lower government consumption, it recovers immediately but remains substantially below the baseline, dropping from 14% to just over 10%. The time-consistent regimes will cut goverment consumption to enable both a drop in taxes that will crowd in private investment, as well as an increase in the share of public investment in public expenditure to support public investment. This allows for higher growth performance. The TP regime initially cuts government consumption but then reverses the policy and after 10 periods has increased government consumption above the baseline. Clearly this policy is not time-consistent, since if the government where to reoptimise in these later periods it would face a very similar incentive to decrease government consumption.

The graph on the level of public investment in the Figure 8.4 shows that, the first few periods apart the public investment share is the virtually the same in both the cooperative and non-cooperative regime. This is an interesting result. I already explained on page 21 that Devereux and Mansoorian (1992) show within their model that the level of public investment would be coordiated efficiently, i.e., that any Nash equilibrium would reach public investment shares that a central planner would choose. My simulations suggest that this result is approximately true in a dynamic setting with time-consistent policy making. The difference between the regimes lies in how the change in government investment is achieved, i.e. which is the mix of the two primary variables. The cooperative solution is to invest relatively more heavily and drive the tax rate lower. The non-cooperative strategy is to keep taxes relatively higher and the share r_t relatively lower. This is the main difference between the regimes. The reason for that is intuitively clear. The government consumption in each country generates felicity for the domestic consumer only. Therefore a government interested in the domestic welfare only will spend more on public consumption than would be desirable from a cooperative time consistent policymakers point of view. That said, we should not loose sight of the optimum policy. Although the cooperative policy is better in the long run, it drives the economy further away from the time-inconsistent optimal path. The non-cooperative policy also delivers a lower "cost-to-go", that is a lower value for utility measured at the starting date. Figures 8.5 and 8.6 show that the lower taxes of the TC regime result in both higher consumption and higher investment by the private sector. Both series remain quite close though and converge over time, leaving only room for small differences in the long run. It is interesting to see how the private sector reacts to the changes on government policy under TP. Since the current tax rate is predetermined, we have $\tau_1 = 0$, then $\tau_2 < 0$, but from then onwards taxes rise and from time 14 onward $\tau_t > 0$, $\forall\, t > 14$. The shift in the infrastructure expenditure from supporting infrastructure to neglecting it occurs in period 15. The time profile of the reaction of the private sector is much smoother then the policy set by the government. Consumption rises immediatelty but returns below the baseline in period 8, well before the reversal in the government's policy, which indicates how important the forward-looking component of consumption is.

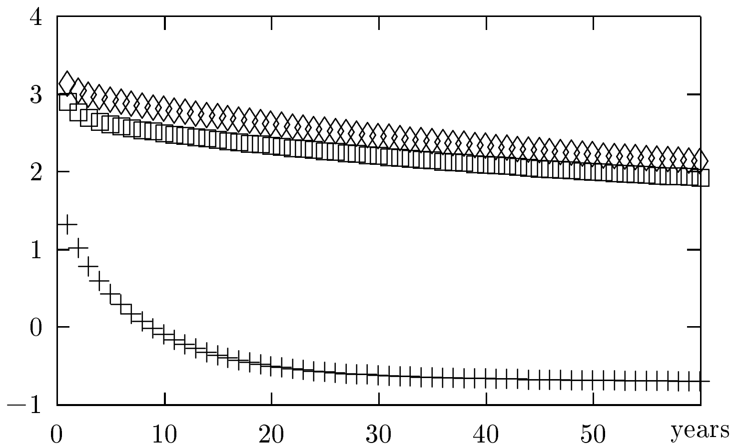

Figure 8.5: c_t, TC ⇜⇝ ◊, TN ⇜⇝ □, TP ⇜⇝ +,

Investment has no forward-looking component, it only depends on next periods' taxes and the current capital stock. In the initial period under all regimes, investment falls. With rising taxation and constant taxes and growth, there is no room in the national expenditure for investment to rise or remain constant. In the time-consistent regime, there is a sustained rise in private investment in the periods that follow. In the inconsistent TP regime, an initial increase in period 2 is reduced over time. However investment only drops below the baseline at period 18, that is after the policy reversal by the government but before the growth rate falls below the baseline. Thus investment reacts with a lag to policy developments.

From the evolution of both private and public investment, we can directly derive the evolution of the corresponding capital stocks. They are pictured in Figure 8.7 and Figure 8.8 respectively. The private capital stock falls in all regimes. That seems

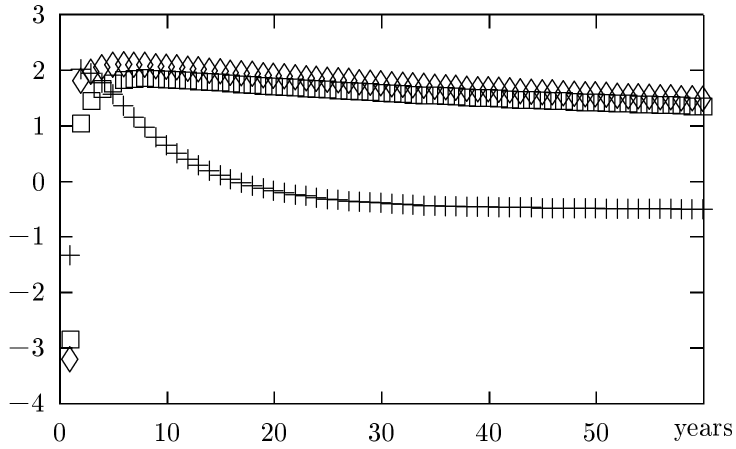

Figure 8.6: i_t, TC $\leftrightsquigarrow \Diamond$, TN $\leftrightsquigarrow \Box$, TP $\leftrightsquigarrow +$,

surprising at first, since Figure 8.6 and 8.4 both suggest increases in investment in the first periods. However there is no conflict between these results because there is an increase in the growth rate. From (5.3) and (5.5) when there is increase in the growth rate, the capital stocks ceteris paribus decline. There is a net shift in the composition of capital from private to public in all regimes; i.e. the GDP share of private capital declines and the share of public capital increases. The shift is stronger for the cooperative regime than for the Nash regime. In the TP regime, the private capital stock per GDP does not fall by as much, but in absolute values it falls by more than in the time-consistent regimes, because of the weak rates of growth. On the other hand, the figures for the time-consistent regime underestimate the capital stock in these regimes because the rate of growth is higher. Consider

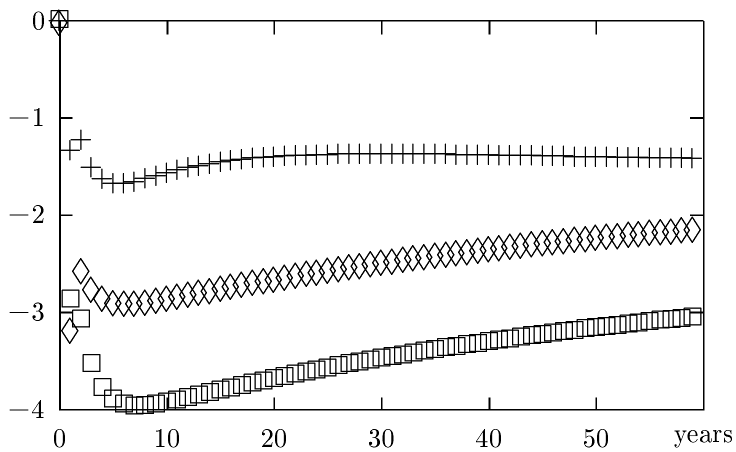

Figure 8.7: k_t, TC $\leftrightsquigarrow \Diamond$, TN $\leftrightsquigarrow \Box$, TP $\leftrightsquigarrow +$,

finally the growth rates and interest rates. In the cooperative regime, growth is higher than in the Nash regime because of the higher infrastructure accumulation and lower taxes. Both time-consistent regimes contrast sharply with the optimal

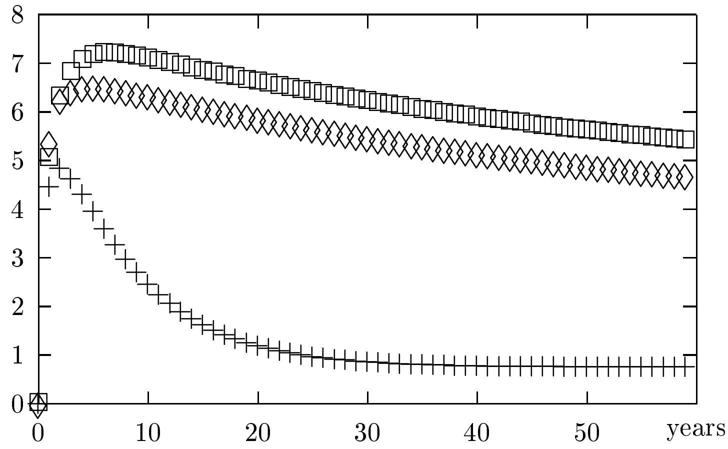

Figure 8.8: k_t^g, TC $\leftrightsquigarrow$ $\Diamond$, TN $\leftrightsquigarrow$ $\Box$, TP $\leftrightsquigarrow$ $+$,

regime. In the second period, growth is larger in TP then in the time-consistent regimes, but from period 4 onwards, growth is slower. Growth returns below the baseline in period 19, and remains there. With a 2% baseline, growth is still positive in the long run, but even a small difference in the long run growth has of course very important implications on long run welfare.

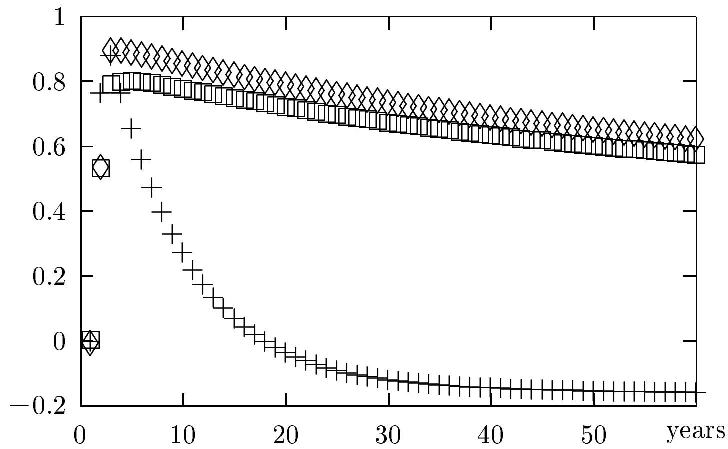

Figure 8.9: n_t, TC $\leftrightsquigarrow$ $\Diamond$, TN $\leftrightsquigarrow$ $\Box$, TP $\leftrightsquigarrow$ $+$,

To summarize, the cooperative time-consistent solution amplifies the difference between the time-consistent and precomitment solution, but not by much. Cooperation is therefore welfare improving in the long run but not on the trajectory.

From the numeric example it should not be concluded that the distinction between optimal and time-consistent regime has more impact on welfare than the distinction between cooperative and non-cooperative fiscal policies. First recall that we do not have values for the non-cooperative reputational regime, and we would need that figure since it may matter a lot under precommitment if the solution is cooperative or not. There is also a problem with the sensitivity of the results. From

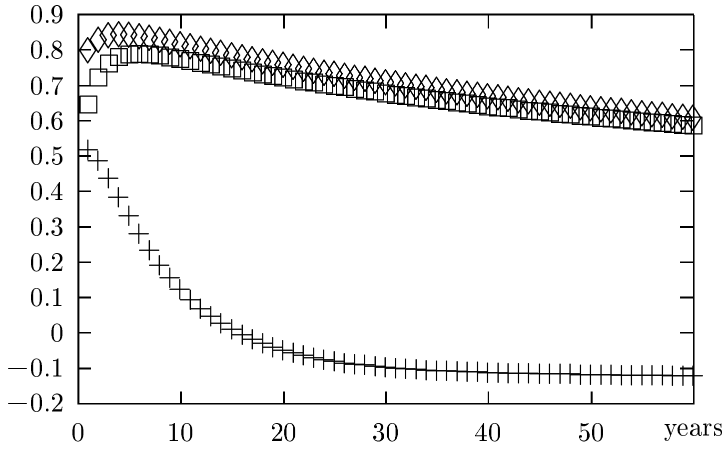

Figure 8.10: r_t, TC ⇝ ◊, TN ⇝ □, TP ⇝ +,

Devereux and Mansoorian (1992) we know that the crucial parameter is the elasticity of substitution $1/\sigma$. They also present simulation evidence suggesting that the minimum externality between policies in different countries can be found at $\sigma \sim 2$. This is of course the central value for the calibration in this model. It should therefore come as no surprise that the difference between the cooperative and the non-cooperative regime is small. I have tried to extent the range of σ. But ACES simulations for the TN regime do not work when σ is outside the set $[1.8, 2.3]$. For values at the margins of that set, there is a large divergence in the values of the long run equilibria. By that I mean that the results are not symmetric, one country is accumulating assets over the other. This should not be the case since the outcome of a symmetric model should be symmetric.

There are two possible explanations for this outcome. One is that the algorithm to compute the solution is correct, but it does not work in this case. Next to nothing is known about the convergence properties of the algorithm. I suspect that the unit root in foreign assets leads to a divergent evolution. The simulation for the first country, leads to a path in which it accumulates so many assets that with given that path, the optimal response of the second government is to accommodate this process. From such an initial situation the algorithm is not likely to converge at all, let alone to a symmetric outcome across countries. There is considerable experimental support for this conjecture. When I replace foreign assets by a variable that is always 0, I do find result for the TN regime as soon as $\sigma \geq 1$.

The problem of non-convergence also affects the TC, the cooperative regime. One would be more comfortable with forcing foreign assets to zero for that regime. One possible way out would be therefore to compare two regimes where we artificially annul-ate the foreign asset accumulation, and hope that the results we have are not too far off the true ones. Unfortunately simulations show that the annulation of foreign assets implies a bias on the results. It changes the comparison of the results, cooperation reduces growth rather than increasing it.

To conclude, in this section I have emphasised the comparison between a cooperative and a non-cooperative equilibrium with time consistent policy. I find that cooperation increases growth. It pushes to the economy further towards the high-growth, low taxation bias that features prominently in the time-consistent case.

9 Debt

Up until now I have assumed that the government budget is in equilibrium at all times. There have only been two policy instruments, global expenditure and the fraction of expenditure spent on infrastructure. Taxes are equal to government spending in these earlier models.

I now enlarge the model to allow taxes to diverge from government spending. This implies the introduction of government debt. The easiest approach to debt is treating it as composed of one-period government bonds. I introduce end-of-period debt as $B(\mathfrak{t})$ and return to the single-country version of the model. Since domestic debt is held by domestic consumers, equation (3.3) becomes

$$Ш^{a}(\mathfrak{t}) = K(\mathfrak{t}) + Л(\mathfrak{t}) + B(\mathfrak{t}) \tag{3.3$'$}$$

The government budget constraint (3.30) becomes

$$B(\mathfrak{t}) = G(\mathfrak{t}) - \tau(\mathfrak{t})\,Q(\mathfrak{t}) + (1 + r(\mathfrak{t}-1))\,B(\mathfrak{t}-1) \tag{3.30$'$}$$

If government expenditure is not bound by a budget constraint—imagine that $B(\mathfrak{t})$ would be free—then the optimal government expenditure is not finite. Therefore we need an additional constraint on government. The most common constraint is solvency, i.e., that the present value of government expenditure minus the present value of its income must be equal to the negative of government debt. The idea is that the tax revenue must be sufficient to both finance expenditure and pay off government debt. Seen in this way, government debt is a predetermined variable that has been determined by the past. In each period the government must respect its budget constraint.

Within the state-space representation of the model it is very difficult to force the government to respect the budget constraint. One can treat debt as a predetermined variable, the evolution of which is given by

$$b_{\mathfrak{t}} = \frac{1+r}{1+n}\,b_{\mathfrak{t}-1} + \frac{b}{1+n}\,r_{\mathfrak{t}-1} - \frac{(1+r)\,b}{(1+n)^2}\,n_{\mathfrak{t}} + g_{\mathfrak{t}} - \tau_{\mathfrak{t}} \tag{9.1}$$

However there is no recipe to ensure government solvency. A simple approach is to penalise the accumulation of debt within the target of the government, i.e., add a term like $\upsilon_b\, b_{\mathfrak{t}}^2$. Unfortunately, under optimal control, the model with debt does not converge unless both changes in debt and changes in the tax rate are subject to stabilisation penalties $\upsilon_{\Delta b}$ and $\upsilon_{\Delta\tau}$ respectively. The time-consistent regime also requests an additional stabilisation penalty on the level of debt υ_b. A problem arises because an arbitrary choice of numbers for both parameters will imply that the government will choose the financial structure of its assets so as to minimize the penalty associated with changes in debt and taxation.

One approach is to depart from a neutrality relationship between the parameters. I can link the parameters in such a way that various financing options will become equivalent as far as the imposition of penalties is concerned. Consider the three following scenarios that all allow to raise one unit of funds:

Policy 1: raise tax this period
Penalty: $(1+\varrho)\,\upsilon_{\Delta\tau}$

This financing option implies changing the tax twice, but no accumulation of debt.

Policy 2: raise debt, pay back next period
Penalty: $(\varrho+\varrho^2)\,(1+r)^2/(1+n)^2\,\upsilon_{\Delta\tau}+(1+\varrho)\,\upsilon_{\Delta b}+\upsilon_b$

This financing option implies no penalty on changing taxes in the first period, but in the two subsequent periods. In addition there is the penalty of changing and holding debt.

Policy 3: raise debt, raise taxes in all period to serve it
Penalty: $\varrho\,(r-n)^2/(1+n)^2\,\upsilon_{\Delta\tau}+\upsilon_{\Delta b}+1/(1-\varrho)\,\upsilon_b$

If I assume that all these policies imply a loss υ, I can deduct

$$\upsilon_{\Delta\tau} = \frac{\upsilon}{1+\varrho} \tag{9.2}$$

$$\upsilon_b = \frac{(1+r)\,(1-\varrho)\,\upsilon}{1+n} \tag{9.3}$$

$$\upsilon_{\Delta b} = \frac{(n-r)\,((1+n)+\varrho\,(1+r))\,\upsilon}{(1+n)^2\,(1+\varrho)} \tag{9.4}$$

The problem with this specification is that the υ term must be set quite high for the programme to converge. After experimentation I chose $\upsilon = 200$. Below that the solution, particularly in the time-consistent regime, exhibits growing fluctuations; they become divergent when the penalty goes below 100, and at about 70 the time consistent procedures does no longer converge, and the Ricatti equation for the optimal control solution can no longer be solved.

If $\upsilon = 200$, the solution is not interesting, since the debt is so heavily penalised that the policy is almost balanced budget. Thus I have not included a further discussion of these results here.

The problem with this approach of a pre-determined government debt is that is neglects the question of solvency. The debt held by the private sector will be paid off over time by taxes levied on the same private sector. Whereas here the private sector includes bonds in its wealth, it does not include the present value of the repayments. This is clearly incompatible with the basic idea of a rational expectations equilibrium.

An alternative view would be to neglect the detail of how government debt is managed and to start with the idea that the private sector own the net worth of the public sector. The net worth of the public sector is the discounted sum of all primary deficits or surpluses from the current period to the infinite future. The net worth is equal to the value of debt if the government is solvent. Therefore I will confuse the notation of the net worth of the government and the stock of bonds, to write

$$B(\mathfrak{t}-1) = \sum_{\mathfrak{t}'=\mathfrak{t}}^{\infty} \frac{\tau(\mathfrak{t})\,Q(\mathfrak{t})-G(\mathfrak{t})}{1+r_{\mathfrak{t}-1}(\mathfrak{t}'-1)} \tag{9.5}$$

where $B(t-1)$ is the net worth of the public sector, which, when the solvency condition holds, is equal to the debt contracted. Define the growth rate between between time t and time t′

$$n_t(t') = \max\left(\prod_{t''=t}^{t'}(1+n(t'')) - 1, 1\right)$$

then in per-GDP terms (9.5) is

$$b(t-1) = \sum_{t'=t}^{\infty} \frac{(1+n_t(t'))\, d(t')}{1+r_{t-1}(t'-1)} \tag{9.6}$$

where I define the primary surplus as $d(t) = \tau(t) - g(t)$. I linearise (9.6) around $b = d = 0$ as

$$b_{t-1} = \sum_{t'=t}^{\infty} \left(\frac{1+n}{1+r}\right)^{t'-t+1} d_{t'} \tag{9.7}$$

or in first difference:

$$\frac{1+n}{1+r}\, b_t = b_{t-1} + g_t - \tau_t \tag{9.8}$$

which is the equation I enter into the software, with b_{t-1} as a free variable. Since wealth includes the stock of debt, the target (5.19) takes proper account of the debt.[23]

[23] Well, almost. There appears a linear term

$$\frac{(1+r)\, b_{t-1}}{(1+\xi)\,(1+n)}$$

This term can be more precise by using the quadratic expansion of b_{t-1}. Since $d = 0$, the only intervening terms are the crossed terms between the interest and growth rates and the surpluses. I will demonstrate the case of the interaction between growth rates and interest rates. Again, let $\mathring{b}_{t-1}$ be the second-order approximation, I have

$$\mathring{b}_t - b_t = \sum_{t'=0}^{\infty} \frac{(1+n)^{t'}}{(1+r)^{t'+1}} d_{t+t'}\, n^{b}_{t+t'} - \sum_{t'=0}^{\infty} \frac{(1+n)^{t'+1}}{(1+r)^{t'+2}} d_{t+t'}\, r^{b}_{t+t'}$$

Drawing heavily on equation (4.12), it can be shown that

$$\sum_{t'=0}^{\infty} \varrho^{t'} \left[\mathring{b}_{t+t'} - b_{t+t'}\right] = \frac{1}{1+n} \sum_{t'=0}^{\infty} \varrho^{t'}\, (t'+1)\, n_{t+t'}\, b_{t+t'-1}$$
$$- \frac{1}{1+r} \sum_{t'=0}^{\infty} \varrho^{t'}\, (t'+1)\, r_{t+t'-1}\, b_{t+t'-1}$$

This is the quadratic expansion of the net worth of the public sector for the crossed terms in growth. In any period, the quadratic approximation of the intertemporal target will take account of all future expressions of this sum. The sequence of terms $\{(t'+1)\, b_{t-1+t'}\}_{t'=0}^{\infty}$ can not be fitted into the state-space representation. An approximation of the sequence would be

$$b^{b}_{t} = b^{b}_{t-1} + b_t \tag{9.9}$$

Let us sum up. My approach is to include the asset position of the public sector as part of the wealth of the private sector. This approach is equivalent to including debt as long as the government is solvent; however its state-space representation is quite different, because the net worth of the public sector is a forward-looking variable.

Treating debt as a non-predetermined variable is of course a controversial assumption. From a mathematical viewpoint, this is nothing else then stating the intertemporal budget constraint of the government. From an economic point of view, it simply expresses that the public own the government and therefore its asset position.

The solvency concept used here completely abstracts from the institutional aspects of solvency. It is not based on any assumption about a long-run value of the government. If the government balances it budget in every period, then its value is zero. If it does not then since it is owned by the private sector, the private sector will, under rational expectations, take account of the value of the state. The value of the state is the discounted sum of the future income minus the future expenditure. I will refer to this concept of solvency as an equity concept of solvency, and the forward looking discounted sum of all future net income stream as the public equity.

There are other concept of solvency, and each brings in an additional level of constraint that. In most cases, researchers require that the public equity will be equal to a certain predetermined sum in each period, usually this sum is taken to be zero. One justification is the idea the debt is financed through some debt instrument. Initially the debt takes a predetermined value. The government can only raise the revenue by raising taxation or selling further debt. I will call this concept the debt concept of solvency. The problem with the debt concept is that it is difficult to implement in a linear-quadratic framework.

There are some refinements of the debt concept of solvency on the market. One is to require that the debt/GDP ratio must be a stable. The advantage of that refinement is that is easy to achieve within the linear quadratic framework; it suffices to penalise debt. Another concept is that debt should be zero in the long run. This

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This alternative term covers the same number of b terms in every period, but not the time profile of these terms. The approximation has the awkward problem that it relies on $b_{t'} = 0, \forall\, t' < t$. The relaxation of the budget constraint in each period to a unique solvency constraint for all period must come as a surprise to the private sector when optimisation starts. For the interest rate, the calculation of a square approximation is equivalent.

To improve the approximation, I could therefore include in the target the quadratic approximation of b_t. This approximation can in turn be approximated using the sum of primary surpluses. The additional term in (5.19) are

$$\frac{c^{-\sigma}}{(1+\xi)\,(1+n)}\, r_{t-1}\, b^{b}_{t-1} - \frac{c^{-\sigma}\,(1+r)}{(1+\xi)\,(1+n)^2}\, n_t\, b^{b}_{t-1}$$

Within ACES, the introduction of a term like b^{b}_{t} requires $b_\infty = 0$ because of the accumulator effect. Thus the strict definition enforces a balanced primary budget and therefore zero public debt. Again I circumvent that feature of the solution by setting the root in (9.9) to 0.99. Unfortunately the introduction of these terms does not give plausible results. All following calculations have been made without them.

is most often used in finite horizon models, there is no justification for it here.

9.1 The results

The results presented here were calculated for a penalty on the change of government spending and on changes of taxation to the tune of 10^{-13} each. Below that value, the calculations for the optimal regime do not converge. For the time-consistent regime it is possible to pick a penalty value as small as 10^{-109}, but the results presented here adopt the same penalty as the optimal regime. Thus any difference between the regimes can not be traced back to different startup penalties. It should however be noted that the time-consistent regime does not converge before 10^6 iterations. It may converge after that but I do not have any evidence at hand. The results here are for 10000 iterations, i.e. the programme calculates 10000 times the next best strategy and then stops, applying the last matrix set that it has found. The results for this regime should therefore be treated with caution. The optimal and time-consistent regimes are labelled DP and DT, respectively. They are single-country, closed-economy simulations.

The long-run results are summarized in Table 9.1. They offer a surprising contrast with the picture that I presented for the balanced-budget regimes. In many ways, the introduction of debt reverses the earlier results. Taxation initially falls, rather than rises, in both regimes but more so in DP rather than in DT. The DP government raises the fraction spent on infrastructure almost to the level that would maximise the steady-state welfare; see the discussion on page 68. Government spending rises under DT, but it falls under DP in the long run. Higher government investment makes for higher growth in the DP regime as compared to the DT regime. On the other hand, government consumption decreases in DP, but rises in DT. All this is quite different from the SP and ST, in fact the comparison appears to be the opposite. It suggests that the results that we found in Section 7 are highly dependent on the inability of governments to raise debt. If it is possible for the precommiting government to raise liabilities, then it will follow a policy that brings it quite close to the policy of the optimal steady state as far as investment is concerned.

The most important aspect to observe here is what I would loosely refer to as the "de Silhouette" property, that is that the value of the state, both in the long run and in the short run is negative. That is, the public, if given a choice, would rather forgo the liabilities of the state. In a conventional model that where government debt is predetermined, there will be negative debt in the longer run. In this model, there will be a higher expenditure than income of the state, which is only be possible if the starting debt is negative. Both approaches are based on the same evolution of debt, but the level of debt is either fixed by a terminal condition or a initial condition.

Note that under the debt concept of solvency, the initial debt would be fixed, and there would be a long-run trend towards negative debt. The profile under time inconsistent policies would to raise taxes a lot in the initial periods, and later relax the tight fiscal policy without bringing about positive debt. Under the time-consistent policy the initial rise in the tax levels would be smaller but the long-run

	DP	DT	SP	ST
b_∞	−62.95	−82.00		
τ_∞	−6.384	−2.648	1.110	−2.913
r_∞	15.482	2.245	−2.272	8.475
r_∞	1.152	0.288	−0.093	0.486
n_∞	1.103	0.920	−0.140	0.484
k_∞	−5.994	4.108	−1.828	−1.666
k^{g}_∞	10.508	−0.341	1.255	3.648
$\text{ш}^{\mathrm{h}}_\infty$	−102.640	−66.542	1.691	−40.823
ξ_∞	−346.011	−170.696	27.811	−145.851
i_∞	2.526	2.644	−0.473	1.165
g^{i}_∞	1.988	1.123	−0.100	0.816
c_∞	1.413	−4.390	−0.637	1.748
g^{c}_∞	−5.927	0.624	1.211	−3.729
ш_∞	−169.139	−199.759	−0.137	−42.490
I_∞	20.520	17.657	−2.757	9.062
G^{i}_∞	10.548	8.265	−1.187	4.573
C_∞	67.729	50.941	−9.057	30.852
G^{c}_∞	9.292	13.322	−0.722	2.950
K_∞	299.895	259.327	−40.663	132.577
K^{g}_∞	156.039	121.083	−17.221	67.516
Ш_∞	−169.139	−210.012	−208.536	677.898
$\text{Ш}^{\mathrm{h}}_\infty$	1232.952	1038.717	−167.873	545.321

Table 9.1: The long run percentage deviations

tax cut would be deeper.

As illustrated in Figure 9.1 the DT regime requires a larger initial surplus, (it generates a larger income/expenditure gap), however the gap reduces over time and after nine periods, it is smaller than the debt under the precommitment regime. However in the long run the liabilities of the DP government are smaller than the ones of the DT regime.

For government spending, Figure 9.2 suggests an important increase in spending in the earlier periods, followed by a decline. From Figure 9.3, the initial rise in spending favours government infrastructure rather than consumption, both regimes implement an infrastructure boom. Note however that government consumption also increases. I suppose this policy is used to counterbalance the effect of the liabilities shock on welfare.

In tax policy, the difference between the two regimes is probably the strongest. As illustrated in Figure 9.4 taxation decreases in both regimes but in the DP regime later reverses that policy. There is a trade-off between stimulating public investment (high spending) and private investment (low taxes). Increasing liabilities is a short-run solution to that problem, but if these liabilities are building up, then there is a

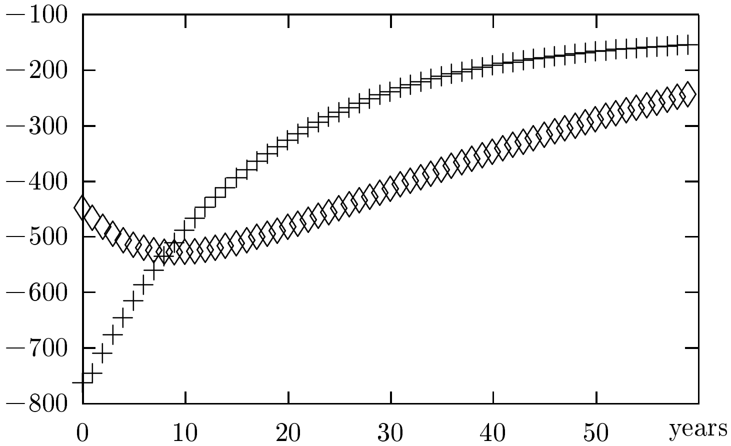

Figure 9.1: b_{t-1}, DP ⇜⇝ ◊, DT ⇜⇝ +,

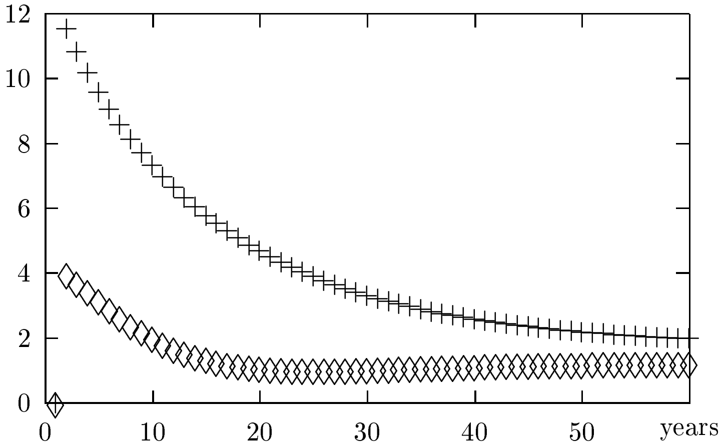

Figure 9.2: spending g_t, DP ⇜⇝ ◊, CT ⇜⇝ +,

negative impact on welfare through the wealth effect that this represents. Therefore in the longer run, the DP government wishes to improve the asset position in periods where the impact that rising taxes and falling spending has on welfare is already heavily discounted. This implies a change in tax policy at period 20. From that period onwards, the impact of further expansionist policies on the economy have less positive impact in the future than negative impact in the present. This intertemporal trade-off implies that that model is stable for an optimal policy. The reversal of policy is of course not time-consistent.

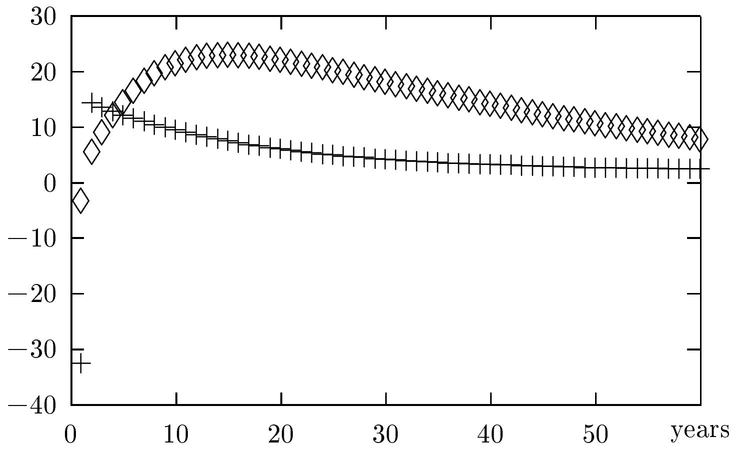

Figure 9.3: r_t, DP ⇜⇝ ◊, CT ⇜⇝ +,

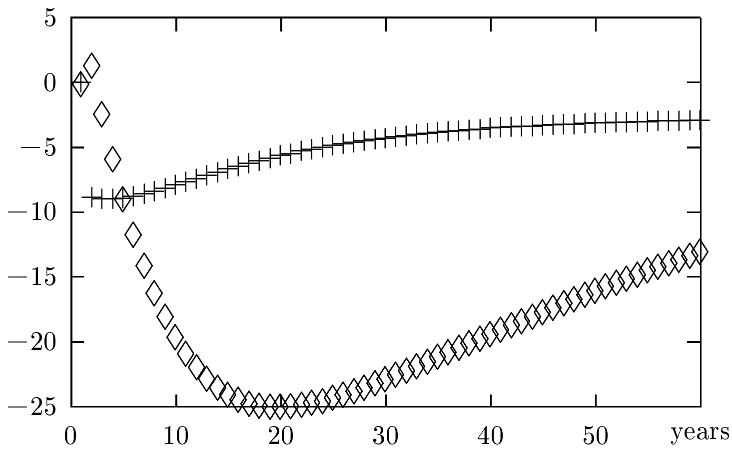

Figure 9.4: taxes τ_t, DP ⇜⇝ ◊, CT ⇜⇝ +,

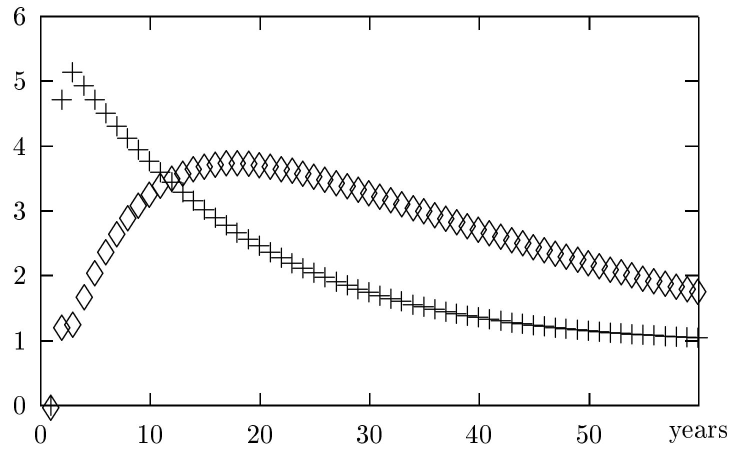

Figure 9.5: growth n_t, DP ⇝ ◊, DT ⇝ +,

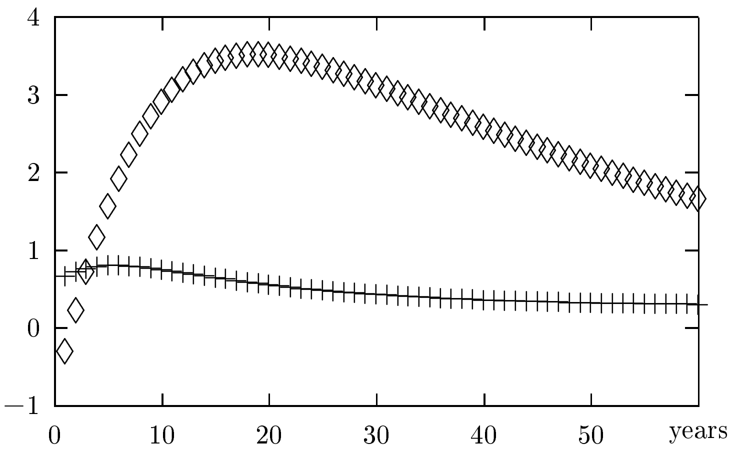

Figure 9.6: interest rate r_t, SP ⇝ ◊, ST ⇝ +,

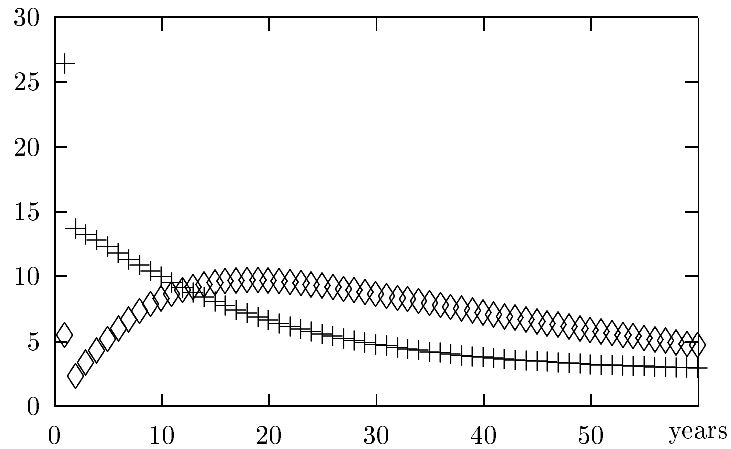

Figure 9.7: Investment i_t, DP ⇝ ◊, DT ⇝ +,

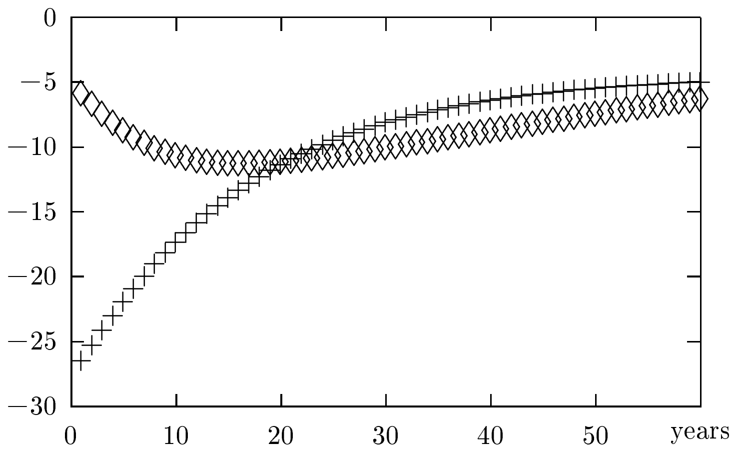

Figure 9.8: consumption c_t, DP ⇜⇝ ◊, CT ⇜⇝ +,

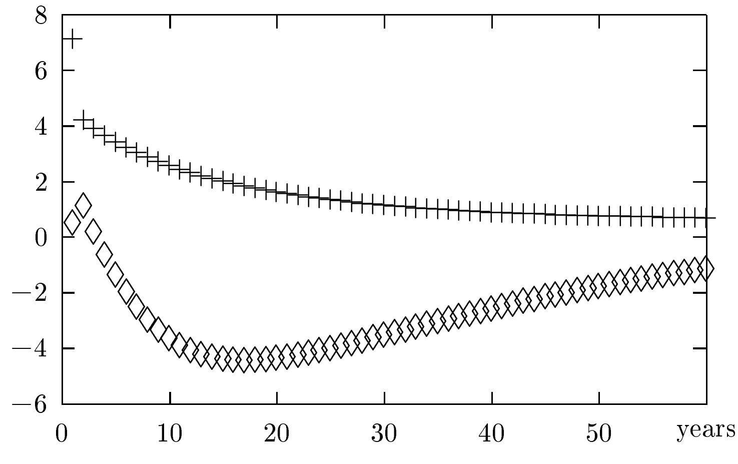

Figure 9.9: public consumption g_t^c, DP ⇜⇝ ◊, CT ⇜⇝ +,

10 Conclusions

The main result of the thesis can be summarized as follows:

- in a model of endogenous growth driven by the accumulation of private and public capital, the long-run welfare is likely to be higher under time-consistent control,

- cooperative control leads to an increase in long-run welfare (when compared to non-cooperative control) through the same approach as the time-consistent control leads to a long-run welfare increase over the time-inconsistent control,

- the increase in long-run welfare of time-consistency and cooperation come at the expense of losses in the cost-to-go

- debt leads to the de Silhouette outcome under the "normal" definitions of solvency.

Let me briefly comment on some issues that surface at several places in this thesis. First consider the theme of endogenous growth. According to some observers the endogenous growth literature is a kind of second birth to macroeconomics, an approach that allows economists to examine the long-run growth of the economy and the factors that might affect it. We have shown here that the way we operate the public decision making process has significant implications on the growth rate. This is a fine example on how the short run and the long run are interrelated and can be simultaneously analysed. There is no longer a devision between the two, nor is there any fixed long-run scenario the short run would needs to converge to etc. Both are solved jointly, with the assumption of rational expectation making the link between the two. From an intellectual point of view this is very satisfying. In many ways the solvency concept used in Section 9 is ideally suited to this concept since it directly integrates the long run consequence of debt into the current consumption function, rather than fixing it to an arbitrary predetermined initial level.

A second important issue is the measurement of social welfare when a time consistency problem arises. Underlying the point that the long-run of the time-consistent is likely to be lead to higher welfare, the whole ethos of optimisation exercises that are conducted in conventional models—where the time consistency problem is usually assumed away. Clearly in this model there is nothing that makes the initially optimum defensible once the first period is over. This is fundamental critique of the conventional optimal control approach to economic policy.

Another interesting angle of this thesis has been the unit root. We show that a simple standard formulation of demand leads to a equilibrium interest rate where assets accumulate as a unit root. There has been a lot of debate on unit roots in the econometric literature, but I am not aware of anyone who would have given a theoretical model where unit roots appear. The theme of the unit root runs through the sections. It appears first in Section 4 where we show that the demand function implies the KL curve. Section 6 then first shows that the unit root is a prominent feature of the model and it then explores where the unit root comes from. Finally the unit root is likely to be at the heart of the problems of convergence of the algorithms in Section 8. I think there is some scope for further work there that would look into conceptualizing that result further and explore it as theory of the origins of fluctuations as an alternative to the two existing paradigms of real business cycles and sunspots.

Now let me consider some of the limitations of the work.

Consider first the subject of growth differentials. Clearly here the most important limitation is that the size of each economy, α, is given. One of the crucial components of a more general model would be the endogenous determination of product varieties, as in the Grossman and Helpman (1992) strand of the literature. In principle, the main idea of this thesis should be unchanged. As long as we can define a price level for each country, and as long as the size of the product spectrum occupied by each economy remains constant in the long run, we can construct a model with growth differentials by relative price adjustments. In fact relative price adjustments not only allows for countries to grow at different rates, they also allow sectors within a country to grow at different rates. This idea was recently discussed by Kongsamut, Rebelo, and Xie (1997).

Another approach to construct a model of endogenous growth with endogenous country size would be to introduce capital accumulation in the Dornbusch, Fisher, and Samuelson (1977)[24] model. This is initially just a Ricardo model where there is no capital at all. To allow for endogenous growth one would need to consider a more general form where human capital and physical capital jointly determine production, and where the comparative costs (the function $a(z)$ of the paper) are determined by the country-specific factor plus perhaps a human capital ingredient. As far as I know, nobody has taken up the challenge.

Another, much simpler variation of the model that I propose in this paper is to model a heterogenous private sector, for example composed out of overlapping generation à la Allais (1947)-Samuelson (1958)-Diamond (1965). It is known since Gale (1971) and Gale (1974) that in this model, trade imbalances can persist in the long run, i.e. there are steady states with imbalance of trade. Overlapping generations would allow for an additional degree of freedom and make the condition for the existence of a steady state with imbalanced growth less stringent. A problem with this approach within the contents of the model here is the indeterminacy of the distribution of income between households of different generations. The conventional approach, pioneered by Diamond (1965) has been to endow the young with a unit of labour, let them work for a period and then buy the capital stock. The old live on the proceeds of the capital stock and supply no labour. It has been shown by Jones and Manuelli (1992) that within that type of scenario, endogenous growth is impossible, because the capital stock becomes so large that the young can no longer purchase it. A simple approach for our purpose here would be to exogenously fix a fraction of income that is owned by the young, but this is rather ad hoc. Another,

[24] Obstfeld and Rogoff (1996) have recently discussed a dynamic version that model.

slightly more Stalinistic approach would be to put all income into the hand of the state and then to allow for the state to distribute income, but then it is not clear why one would have distortionary taxes in the model as well.

For the time-consistency issues, the most important limit of the work here is the linear-quadratic framework. This poses two problems. On a technical level it requires each model to be of a linear quadratic form. With each new non-linear model we need first to derive the linear-quadratic approximation. The linearisation of the model itself is not a problem, but the quadratification of utility is a serious challenge for an economist. One important message that I learned from doing it is that the consequences of getting things wrong here are usually that the computer programme does not converge, or only converges under heavy instrument penalties etc. Only if the quadratic utility function is *exactly* right then the computer programme will run and give meaningful results. In this thesis I use a really simple model, for more complicated models the calculation of the linear-quadratic approximation would be practically infeasible. I am not sure how it could be done on a computer, given the fact that it involves an infinite time horizon.

Another serious obstacle is the problem of multiple solutions. Models of the genre considered here in this thesis can have multiple long run equilibria. The problem with calculating these equilibria is the presence of forward-looking variables like consumption. To know consumption we need to know the future states of the system. But these again depend on the variables being set today etc. Linearisation avoids that catch-22 situation. It arbitrary fixes a long run and provided that the model is stable, it will return to that long run. Since the full model behaves like the linear model only for an infinitely small displacement, we know nothing about the behaviour of the non-linear model for any finite sized shock. There is no reason why a model that has been displaced from an initial steady state should go back to it. There is nothing wrong with assuming that it does. However since we need to know the long run trajectory, we can only use a linear approximation to calculate the transition of the model from its displaced state to the assumed steady state.

If the calculation of the trajectory for an arbitrary shock cannot be done for a full non-linear model, it is even more impossible to calculate a time-consistent policy. I suspect that the non-linear features of the model do in fact stabilise the model, although I have nothing to back up that conviction. In the particular model that I look at here, one important feature seems to me that there is an optimal steady state; this is a steady state such that a reoptimisation departing from this steady state does not yield any further utility gain. I have not formally proven that here. But if this property holds then we do know that the model does have a stable optimum long run and that once the policymaker reaches this long run it will stay there. For this case—I think—it should be possible to calculate an approximation to the time-consistent trajectory that will lead to this long-run steady state.

We know that the time-consistent policy is a function $f(\cdot)$ that maps the current state $\mathbf{y}(t)$ of the system into the policy variables $\mathbf{w}(t)$. In a first stage, we can calculate the optimal steady state, by considering the static problem of picking the optimal steady state out of all steady states, regardless of transitory dynamics.

This leads to an optimal steady state $\tilde{\mathbf{z}}$, with an instrument choice of $\tilde{\mathbf{w}}$. Once the government has reached the steady state say $\tilde{\mathbf{y}}(\mathbf{t})$, it can not improve on it. This imposes the restriction $\tilde{\mathbf{w}} = f(\tilde{\mathbf{y}})$. One possibility is of course to set a functional form for $f(\cdot)$, and then calculate the optimal solution under the constraint of the functional form. Given the number of functional forms and variables, this should be difficult. Any solution would be strictly time-consistent, but would only be optimal for the particular class of policy functions under consideration. Clearly the choice of a functional form of $f(\cdot)$ has important implications on the form that the adjustment path will take.

Continue to assume that the model has an optimal steady state. That is a steady-state which the government would not move away from. Then we know that the government would like to reach that steady state in the long run, the only problem is that it can not reach it immediately or that reaching it immediately may be costly. We can then try to iterate over the number of periods that it takes the government to reach the steady state, and with a large number of periods we may be able to closely approximate this policy[25]. Maybe, in my next life, I will try to calculate that iteration.

[25] More formally, consider a model with transitory dynamics, where the state is $\mathbf{y}(\mathbf{t}) = [\mathbf{z}(\mathbf{t}), \mathbf{x}(\mathbf{t})]$, where $\mathbf{z}$ is a vector if predetermined variable and $\mathbf{x}$ a vector of free variables. Then the model is represented as $\mathbf{y}(\mathbf{t}) = \mathbf{m}(\mathbf{z}(\mathbf{t}-1), \mathbf{x}(\mathbf{t}-1), \mathbf{w}(\mathbf{t}))$, where $\mathbf{x}(\mathbf{t}-1) = \mathbf{g}(\mathbf{y}(\mathbf{t}), \mathbf{y}(\mathbf{t}+1), \ldots, \mathbf{w}(\mathbf{t}), \mathbf{w}(\mathbf{t}+1), \ldots)$.

One possible policy is to pass immediately from $\mathbf{y}(0)$ to $\tilde{\mathbf{y}}$. This policy should be feasible if $\mathbf{y}(0)$ is not too far from $\tilde{\mathbf{y}}$. In this setting we have completely characterized the state of the system. We know that $\mathbf{y}$ takes the values $\mathbf{y}(0), \tilde{\mathbf{y}}, \tilde{\mathbf{y}}, \ldots$. That allows us to compute $\mathbf{x}(\mathbf{t}) = \mathbf{g}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}, \ldots, \mathbf{w}(1), \tilde{\mathbf{w}}, \tilde{\mathbf{w}}, \ldots)$. This will then allow to calculate $\mathbf{w}(1)$ as a function of $\mathbf{y}(0)$.

Then allow the government to get to the optimal steady state in two periods. Here the trajectory is $\mathbf{y}(0), \mathbf{y}(1), \tilde{\mathbf{y}}, \tilde{\mathbf{y}}, \ldots)$. Since we have calculated welfare from period 2 onwards as a function of $\mathbf{y}(1)$, the optimal solution is found by minimising the discounted utility over both periods. This defines an iterative procedure to find the optimal policy. These iterations will not lead to a time consistent policy because a later government can always reexamine the length of the interval by which it will converge. However if the number of iterations is large enough, one should hope that one approaches the time-consistent policy.

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A The Solution Procedures in a Single Country

A.1 Setting up the linear version

The model of Section 5.1 can be expressed in state-space form as

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{x}^{e}_{t+1,t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{z}_{t} \\ \mathbf{x}_{t} \end{bmatrix} + \mathbf{B}\,\mathbf{w}_t \tag{A.1}$$

$$\mathbf{s}_t = \mathbf{E}_1 \begin{bmatrix} \mathbf{z}_{t} \\ \mathbf{x}_{t} \end{bmatrix} + \mathbf{E}_2\,\mathbf{w}_t \tag{A.2}$$

where $\mathbf{z}_t$ is an $n_p \times 1$ vector of predetermined variables at time t. $\mathbf{x}_t$ is an $n_f \times 1$ vector of free variables, and $\mathbf{x}^e_{t+1,t}$ denotes rational expectations of $\mathbf{x}_{t+1}$. In our model there are only three non-predetermined variables, consumption and the two forward-looking variables for government policy, therefore $n_f = 3$. $\mathbf{s}_t$ is an $n_t \times 1$ vector of target variables, expressed as deviation from a bliss point. In our model the bliss point for consumption and government consumption are 100% of GDP, and the bliss point for growth is 100% as well. Of course these points can not be achieved simultaneously at any date, therefore the welfare loss will be strictly positive. The loss of the government is written as

$$U_t = \frac{1}{2}\sum_{t'=0}^{\infty} \varrho^{t'}\,\mathbf{s}^{\top}_{t+t'}\,\Upsilon\,\mathbf{s}_{t+t'} \tag{A.3}$$

where Υ is a symmetric and positive definite matrix of weights and $\varrho > 0$ is the discount factor. The policymaker's optimization problem is to minimize U_t subject to the model (A.1) and the initial vector $\mathbf{z}_t$. Substituting (A.2) in (A.3) will give the following form of the welfare loss

$$U_t = \frac{1}{2}\sum_{t'=0}^{\infty} \varrho^{t'}\left[\mathbf{y}^{\top}_{t+t'}\,\mathbf{Q}\,\mathbf{y}_{t+t'} + 2\,\mathbf{y}^{\top}_{t+t'}\,\mathbf{U}\,\mathbf{w}_{t+t'} + \mathbf{w}^{\top}_{t+t'}\,\mathbf{R}\,\mathbf{w}_{t+t'}\right] \tag{A.4}$$

Where we use the definitions $\mathbf{Q} = \mathbf{E}_1^{\top}\,\Upsilon\,\mathbf{E}_1$, $\mathbf{U} = \mathbf{E}_1^{\top}\,\Upsilon\,\mathbf{E}_2$, and $\mathbf{R} = \mathbf{E}_2^{\top}\,\Upsilon\,\mathbf{E}_2$. We also introduce the notation $\mathbf{y}_t^{\top} = [\mathbf{z}_t^{\top}, \mathbf{x}_t^{\top}]$ as the state vector, of dimension $n_s \times 1$, where $n_s = n_p + n_f$. For the vectors that have the dimension $n_s \times 1$, it is convenient to partition the vector into the first n_p elements and the n_f elements that follow. Using this notation, for example

$$\mathbf{y}_t \equiv \begin{bmatrix} \mathbf{y}_{p,t} \\ \mathbf{y}_{f,t} \end{bmatrix} \tag{A.5}$$

where here of course $\mathbf{y}_{p,t} = \mathbf{z}_t$ and $\mathbf{y}_p = \mathbf{z}_t$. It is also inconvenient to introduce a similar notation for matrices. Let $\mathbf{X}$ be any matrix of dimension $n_s \times n_s$, then write

$$\mathbf{X} \equiv \begin{bmatrix} \mathbf{X}_{p,p} & \mathbf{X}_{p,f} \\ \mathbf{X}_{f,p} & \mathbf{X}_{f,f} \end{bmatrix} \tag{A.6}$$

such that $\mathbf{X}_{p,p}$ is of dimension $n_p \times n_p$ $\mathbf{X}_{f,p}$ is of dimension $n_f \times n_p$ $\mathbf{X}_{p,f}$ is of dimension $n_p \times n_f$ and $\mathbf{X}_{f,f}$ is of dimension $n_f \times n_f$. We will make repeated use of this notation in the remainder of the appendix, when we develop the solution procedures for both the precommitment and the time consistent case.

A.2 The optimal policy with precommitment

To find the optimum policy under precommitment, consider the government's ex-ante optimum policy at $t = 0$ under the assumption that precommitment is possible. By standard theory of Lagrangian multipliers, we then minimize the Lagrangian

$$\mathcal{L}_0 = U_0 + \sum_{t=0}^{\infty} \varrho^t \boldsymbol{\lambda}_t \left[\mathbf{A}\,\mathbf{y}_t + \mathbf{B}\,\mathbf{w}_t - \mathbf{y}_{t+1} \right] \tag{A.7}$$

with respect to $\{\mathbf{y}_t\}_{t=0}^{\infty}$, $\{\boldsymbol{\lambda}_t\}_{t=0}^{\infty}$, and $\{\mathbf{w}_t\}_{t=0}^{\infty}$, for a given $\mathbf{z}_0$. This gives the first order conditions that

$$\mathbf{w}_t = -\mathbf{R}^{-1} \left[\varrho\,\mathbf{B}^\top \boldsymbol{\lambda}_{t+1} + \mathbf{U}^\top \mathbf{y}_t \right] \tag{A.8}$$

$$\mathbf{U}\,\mathbf{w}_t = \boldsymbol{\lambda}_t - \varrho\,\mathbf{A}^\top \boldsymbol{\lambda}_{t+1} - \mathbf{Q}\,\mathbf{y}_t \tag{A.9}$$

together with the original constraint

$$\mathbf{y}_{t+1} = \mathbf{A}\,\mathbf{y}_t + \mathbf{B}\,\mathbf{w}_t \tag{A.10}$$

Equations (A.8), (A.9) and (A.10) hold for $t \geq 1$. They can be written in state-space form as

$$\begin{bmatrix} \mathbf{I} & \varrho\,\mathbf{B}\,\mathbf{R}^{-1}\,\mathbf{B}^\top \\ \mathbf{0} & \varrho\,(\mathbf{A}^\top - \mathbf{U}\,\mathbf{R}^{-1}\mathbf{B}^\top) \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t+1} \\ \boldsymbol{\lambda}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\,\mathbf{R}^{-1}\,\mathbf{U}^\top & \mathbf{0} \\ -\mathbf{Q} + \mathbf{U}\,\mathbf{R}^{-1}\,\mathbf{U}^\top & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} \tag{A.11}$$

The solution to (A.11) requires $2\,n_s$ boundary conditions. The first order condition in $t = 0$, requires that

$$\boldsymbol{\lambda}_0^\top\,\mathrm{d}\,\mathbf{y}_0 = 0 \tag{A.12}$$

Within $\mathbf{y}_0$ the first n_p elements are predetermined, therefore $\mathrm{d}\mathbf{y}_0^p = 0$, whereas the n_f elements that follow are free and therefore require from (A.12) that

$$\boldsymbol{\lambda}_{f,0} = \mathbf{0} \tag{A.13}$$

This gives n_f boundary conditions to solve (A.11). The initial value $\mathbf{z}_0$ gives n_p more conditions. Finally the transversality condition

$$\lim_{t \to \infty} \varrho^t\,\boldsymbol{\lambda}_t = \mathbf{0} \tag{A.14}$$

provides n_s more conditions, which complete to the required $2\,n_s$ boundary conditions. The solution takes the form

$$\boldsymbol{\lambda}_t = \mathbf{S}\,\mathbf{y}_t \tag{A.15}$$

Substituting into (A.9) we get

$$\begin{aligned} \mathbf{w}_t &= -\left(\mathbf{R} + \mathbf{B}^\top \mathbf{S}\,\mathbf{B}\right)^{-1} \left(\mathbf{B}^\top \mathbf{S}\,\mathbf{A} + \mathbf{U}^\top\right) \mathbf{y}_t \\ &= -\mathbf{F}\,\mathbf{y}_t \end{aligned} \tag{A.16}$$

say, where $\mathbf{S}$ is the solution to the Ricatti matrix equation

$$\begin{aligned}\mathbf{S} = \mathbf{Q} - \mathbf{U}\,\mathbf{F} - \mathbf{F}^\top \mathbf{U}^\top + \mathbf{F}^\top \mathbf{R}\,\mathbf{F} \\ + (\mathbf{A} - \varrho\,\mathbf{B}\,\mathbf{F})^\top \mathbf{S}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F})\end{aligned} \tag{A.17}$$

All that is not left to complete the solution is to express the non-predetermined variables at time t, $\begin{bmatrix}\boldsymbol{\lambda}_{p,t}^\top & \boldsymbol{\lambda}_{p,t}^\top\end{bmatrix}^\top$ in terms of the predetermined variables $\begin{bmatrix}\mathbf{z}_t^\top & \boldsymbol{\lambda}_{p,t}^\top\end{bmatrix}^\top$. Rearranging (A.15), we obtain

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\lambda}_{p,t} \\ \mathbf{x}_t\end{bmatrix} &= \begin{bmatrix}\mathbf{S}_{p,p} - \mathbf{S}_{f,f}^{-1}\,\mathbf{S}_{f,p} & \mathbf{S}_{p,f}\,\mathbf{S}_{f,f}^{-1} \\ -\mathbf{S}_{f,f}^{-1}\,\mathbf{S}_{f,p} & \mathbf{S}_{f,f}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix} \\ &= -\mathbf{N}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix}\end{aligned} \tag{A.18}$$

say. Substituting into (A.16) gives

$$\begin{aligned}\mathbf{w}_t &= -\mathbf{F}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ -\mathbf{N}_{f,p} & -\mathbf{N}_{f,f}\end{bmatrix}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix} \\ &= \mathbf{G}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix}\end{aligned} \tag{A.19}$$

say, and combining (A.10), (A.16) and (A.18) gives

$$\begin{aligned}\begin{bmatrix}\mathbf{z}_{t+1} \\ \boldsymbol{\lambda}_{p,t+1}\end{bmatrix} &= \mathbf{T}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F})\,\mathbf{T}^{-1}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix} \qquad \text{where} \quad \mathbf{T} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{S}_{f,p} & \mathbf{S}_{f,f}\end{bmatrix} \\ &= \mathbf{H}\begin{bmatrix}\mathbf{z}_t \\ \boldsymbol{\lambda}_{p,t}\end{bmatrix}\end{aligned} \tag{A.20}$$

say. Given the solution $\mathbf{S}$ to the Ricatti equation (A.17), equations (A.18) to (A.20) completely characterize the solution to the optimization problem. The solution can be expressed as a feedback on the history of the state vectors. At $t = 0$, this feedback is simply given by (A.19). To find the feedback for the following periods, use (A.20) to write

$$\boldsymbol{\lambda}_{p,t+1} = \mathbf{H}_{f,p}\,\mathbf{z}_t + \mathbf{H}_{f,f}\,\boldsymbol{\lambda}_{p,t} \tag{A.21}$$

Solving (A.21) and using (A.12), we find

$$\boldsymbol{\lambda}_{p,t+1} = \mathbf{H}_{f,p}\sum_{t'=0}^{t}(\mathbf{H}_{f,f})^{t'}\mathbf{z}_{t-t'} \tag{A.22}$$

Hence the feedback form of the rule $\mathbf{w}_t = \mathbf{G}_p\,\mathbf{z}_t + \mathbf{G}_f\,\boldsymbol{\lambda}_{f,t}$ can be expressed solely in terms of the (at time t) predetermined variables $\mathbf{z}_t$.

Finally let us evaluate the welfare loss along the trajectory or "cost-to-go". From the envelope theorem and the first order condition (A.12), we have that

$$\frac{\mathrm{d}U}{\mathrm{d}\mathbf{y}_0} = \frac{\mathrm{d}\mathcal{L}_0}{\mathrm{d}\mathbf{y}_0} = \boldsymbol{\lambda}_0^\top \tag{A.23}$$

Hence from (A.15) on integration we have

$$U_0 = \frac{1}{2} \mathbf{y}_0^\top \mathbf{S} \mathbf{y}_0 \tag{A.24}$$

at time $t' = 0$. At time t this becomes

$$U_t = \frac{1}{2} \mathbf{y}_t^\top \mathbf{S} \mathbf{y}_t \tag{A.25}$$

Another way of expressing U_t, which will proof useful, is found by eliminating $\mathbf{x}_t$ in (A.25) using (A.18). We obtain

$$U_t = -\frac{1}{2}\left[\text{trace}(\mathbf{N}_{p,p}\, \mathbf{z}_t\, \mathbf{z}_t^\top) + \text{trace}(\mathbf{N}_{f,f}\, \boldsymbol{\lambda}_{p,t}\, \boldsymbol{\lambda}_{p,t}^\top)\right] \tag{A.26}$$

which at $t = 0$, using (A.13) becomes

$$U_0 = -\frac{1}{2}\left[\text{trace}(\mathbf{N}_{p,p}\, \mathbf{z}_0\, \mathbf{z}_0^\top)\right] \tag{A.27}$$

A.3 The time consistent (Markov-perfect) solution

The precommitment solutions takes the feedback form of a rule (A.19) which as we have seen from (A.21) is a rule with memory. The time-inconsistency of this solution is best seen by examining the cost-to-go (A.25). Re-optimising at time t and reneging on the commitment given at time 0 involves putting $\boldsymbol{\lambda}_{p,t} = \mathbf{0}$. Thus the gains from reneging are $-\text{trace}(\mathbf{N}_{f,f}\, \boldsymbol{\lambda}_{p,t}\, \boldsymbol{\lambda}_{p,t}^\top)$. Since it can be shown that $\mathbf{N}_{f,f}$ is negative definite (Currie and Levine (1994), chapter 5, page 145 for a formal proof), it follows that everywhere along the trajectory at which $\boldsymbol{\lambda}_{f,t} \neq \mathbf{0}$ there will be gains from reneging and the ex ante optimal policy will be suboptimal ex post.

In order to construct a time-consistent policy we employ dynamic programming and seek a Markov-perfect equilibrium in which instruments are still allowed to depend on the past history, but only through a feedback on the *current* value of the state variables. This precludes feedback as in (A.21) which involves memory. Thus we seek a stationary solution $\mathbf{w}_t = \mathbf{G}\, \mathbf{z}_t$ in which U_t is minimized at the time t subject to the model (A.2) in the knowledge that an identical procedure will be used to determine U_{t+1} at time $t+1$. Other features of the solution are the $\mathbf{x}_{t'} = -\mathbf{N}\, \mathbf{z}_{t'}$, which we know is true of saddle-path stable solutions to rational expectations models under a rule $\mathbf{w}_{t'} = -\mathbf{F}\, \mathbf{z}_{t'}$, and $U_t = \mathbf{z}_t^\top \mathbf{S}\, \mathbf{z}_t$. Notice that all three solution features follow from the precommitment solution with $\boldsymbol{\lambda}_{p,t} = \mathbf{0}$ for all t. The solution is completely characterized by the matrices $\mathbf{F}$, $\mathbf{N}$ and $\mathbf{S}$. We now derive an iterative procedure and sequences $\mathbf{F}_t$, $\mathbf{N}_t$ and $\mathbf{S}_t$ which—if convergent—converge to the these stationary values. Suppose that from time $t+1$ onwards,

$$\mathbf{x}_{t+t'} = -\mathbf{N}_{t+1}\, \mathbf{z}_{t+t'} \qquad \forall\, t' \geq 1 \tag{A.28}$$

Then from (A.1)

$$\begin{aligned} \mathbf{x}_{t+1} &= -\mathbf{N}_{t+1}\left(\mathbf{A}_{p,p}\, \mathbf{z}_t + \mathbf{A}_{p,f}\, \mathbf{x}_t + \mathbf{B}_p\, \mathbf{w}_t\right) \\ &= \mathbf{A}_{f,p}\, \mathbf{z}_t + \mathbf{A}_{f,f}\, \mathbf{x}_t + \mathbf{B}_f\, \mathbf{w}_t \end{aligned} \tag{A.29}$$

Thus

$$\mathbf{x}_t = \mathbf{J}_t\,\mathbf{z}_t + \mathbf{K}_t\,\mathbf{w}_t \tag{A.30}$$

where

$$\mathbf{J}_t = -(\mathbf{A}_{f,f} + \mathbf{N}_{t+1}\,\mathbf{A}_{p,f})^{-1}\,(\mathbf{N}_{t+1}\,\mathbf{A}_{p,p} + \mathbf{A}_{f,p}) \tag{A.31}$$

$$\mathbf{K}_t = -(\mathbf{A}_{f,f} + \mathbf{N}_{t+1}\,\mathbf{A}_{p,f})^{-1}\,(\mathbf{N}_{t+1}\,\mathbf{B}_p + \mathbf{B}_f) \tag{A.32}$$

Rewrite (A.4) as

$$U_t = \frac{1}{2}\,(\mathbf{y}_t^\top\,\mathbf{Q}\,\mathbf{y}_t + 2\,\mathbf{y}_t^\top\,\mathbf{U}\,\mathbf{w}_t + \mathbf{w}_t^\top\,\mathbf{R}\,\mathbf{w}_t) + \varrho\,U_{t+1} \tag{A.33}$$

then putting $U_{t+1} = \mathbf{z}_{t+1}^\top\,\mathbf{S}_{t+1}\,\mathbf{z}_{t+1}/2$, and substituting for $\mathbf{x}_t$ from (A.30), we obtain

$$U_t = \frac{1}{2}\left(\mathbf{z}_t^\top\,\bar{\mathbf{Q}}_t\,\mathbf{z}_t + 2\,\mathbf{z}_t^\top\,\bar{\mathbf{U}}_t\,\mathbf{w}_t + \mathbf{w}_t^\top\,\bar{\mathbf{R}}_t\,\mathbf{w}_t\right) + \frac{\varrho\,\mathbf{z}_{t+1}^\top\,\mathbf{S}_{t+1}\,\mathbf{z}_{t+1}}{2} \tag{A.34}$$

where

$$\bar{\mathbf{Q}}_t = \mathbf{Q}_{p,p} + \mathbf{J}_t^\top\,\mathbf{Q}_{f,p} + \mathbf{Q}_{p,f}\,\mathbf{J}_t + \mathbf{J}_t^\top\,\mathbf{Q}_{f,f}\,\mathbf{J}_t \tag{A.35}$$

$$\bar{\mathbf{U}}_t = \mathbf{U}_p + \mathbf{Q}_{p,f}\,\mathbf{K}_t + \mathbf{J}_t^\top\,\mathbf{U}_f + \mathbf{J}_t^\top\,\mathbf{Q}_{f,f}\,\mathbf{J}_t \tag{A.36}$$

$$\bar{\mathbf{R}}_t = \mathbf{R} + \mathbf{U}_f^\top\,\mathbf{K}_t + \mathbf{K}_t^\top\,\mathbf{U}_f + \mathbf{K}_t^\top\,\mathbf{Q}_{f,f}\,\mathbf{K}_t \tag{A.37}$$

Similarly eliminate $\mathbf{x}_t$ from (A.1) to obtain

$$\mathbf{z}_{t+1} = \bar{\mathbf{A}}_t\,\mathbf{z}_t + \bar{\mathbf{B}}_t\,\mathbf{w}_t \tag{A.38}$$

where

$$\bar{\mathbf{A}}_t = \mathbf{A}_{p,p} + \mathbf{A}_{p,f}\,\mathbf{J}_t \tag{A.39}$$

$$\bar{\mathbf{B}}_t = \mathbf{B}_p + \mathbf{A}_{p,f}\,\mathbf{K}_t \tag{A.40}$$

Hence substituting (A.38) into (A.34) we arrive at

$$\begin{aligned} U_t = \frac{1}{2}\,\big[&\mathbf{z}_t^\top(\bar{\mathbf{Q}}_t + \varrho\,\bar{\mathbf{A}}_t\,\mathbf{S}_{t+1}\,\bar{\mathbf{A}}_t)\,\mathbf{z}_t \\ &+ 2\,\mathbf{z}_t^\top(\bar{\mathbf{U}}_t + \varrho\,\bar{\mathbf{A}}_t^\top\mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{w}_t \\ &+ \mathbf{w}_t^\top(\bar{\mathbf{R}}_t + \varrho\,\bar{\mathbf{B}}_t^\top\,\mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{w}_t\big] \end{aligned} \tag{A.41}$$

The control problem is now to minimize U_t with respect to $\mathbf{w}_t$ given the current state $\mathbf{z}_t$. and given $\mathbf{S}_{t+1}$ and $\mathbf{N}_{t+1}$ which are determined by subsequent reoptimisation. The first order condition is then

$$\begin{aligned} \mathbf{w}_t &= (\bar{\mathbf{R}}_t + \varrho\,\bar{\mathbf{B}}_t^\top\,\mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)^{-1}(\bar{\mathbf{U}}_t^\top + \varrho\,\bar{\mathbf{A}}_t^\top\,\mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,\mathbf{z}_t \\ &= \mathbf{G}_t\,\mathbf{z}_t \end{aligned} \tag{A.42}$$

say. Then combining (A.30) and (A.42) we have

$$\mathbf{x}_t = (\mathbf{J}_t - \mathbf{K}_t\,\mathbf{G}_t)\,\mathbf{z}_t \tag{A.43}$$

$$= -\mathbf{N}_t\,\mathbf{z}_t \tag{A.44}$$

say. Substituting (A.42) into (A.41) and equating the quadratic terms in $\mathbf{z}_t$ gives

$$\begin{aligned}\mathbf{S}_t = \bar{\mathbf{Q}}_t + \bar{\mathbf{U}}_t\,\mathbf{G}_t + \mathbf{G}_t^\top\,\bar{\mathbf{U}}_t^\top + \mathbf{G}_t^\top\,\bar{\mathbf{R}}_t\,\mathbf{G}_t \\ + (\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}_t)^\top\mathbf{S}_{t+1}(\varrho\,\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}_t)\end{aligned} \tag{A.45}$$

Given $\mathbf{S}_{t+1}$ and $\mathbf{N}_{t+1}$ equations (A.42), (A.43) and (A.45) give $\mathbf{F}_t$, $\mathbf{N}_t$, and $\mathbf{S}_t$ defining our iterative process. If these converge[26] to stationary values $\mathbf{F}$, $\mathbf{N}$ and $\mathbf{S}$, then we have a time-consistent optimal rule $\mathbf{w}_t = \mathbf{G}\,\mathbf{z}_t$, with cost to go

$$U_t = \frac{1}{2}\,\mathbf{z}_t^\top\,\mathbf{S}\,\mathbf{z}_t = \frac{1}{2}\,\mathrm{trace}(\mathbf{S}\,\mathbf{Z}_t) \tag{A.46}$$

[26] We have not found any problems with convergence for a wide range of models, including that in this paper

B The Solution Procedures for the two country game

Replace the subscription t in Section 5.1 with $t+1$ and take expectations at time t. The model can then be expressed in state-space form as

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{x}^e_{t+1,t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + \mathbf{B} \begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}^*_t \end{bmatrix} \tag{B.1}$$

$$\mathbf{s}_t = \mathbf{E}_1 \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + \mathbf{E}_2 \begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}^*_t \end{bmatrix} \tag{B.2}$$

$$\mathbf{s}^*_t = \mathbf{E}^*_1 \begin{bmatrix} \mathbf{z}_t \\ \mathbf{x}_t \end{bmatrix} + \mathbf{E}^*_2 \begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}^*_t \end{bmatrix} \tag{B.3}$$

where $\mathbf{z}_t$ is an $n_p \times 1$ vector of predetermined variables at time t, $\mathbf{x}_t$ is an $n_f \times 1$ vector of free variables, and $\mathbf{x}^e_{t+1,t}$ denotes rational expectations of $\mathbf{x}_{t+1}$ formed at time t on the basis of the information set $\{\mathbf{z}_{t'}, \mathbf{x}_{t'}, t+1 \leq t\}$. $\mathbf{w}_t$ and $\mathbf{w}^*_t$ are $n_i \times 1$ vectors of instruments available to the domestic and foreign governments, respectively. $\mathbf{s}_t$ and $\mathbf{s}^*_t$ are $n_t \times 1$ vectors of target variables for the domestic and foreign government, respectively, both expressed as deviation from a bliss point.

The loss of the domestic government is written as

$$U_t = \frac{1}{2} \sum_{t'=0}^{\infty} \varrho^{t'}\, \mathbf{s}^\top_{t+t'}\, \Upsilon\, \mathbf{s}_{t+t'} \tag{B.4}$$

where Υ is a symmetric and positive definite matrix of weights and $\varrho > 0$ is the discount factor. The policymaker's optimization problem is to minimize U_t subject to the model (B.1) and the initial vector $\mathbf{z}_t$. Substituting (B.2) in (B.4) will give the following form of the welfare loss

$$\begin{aligned} U_t = \frac{1}{2} \sum_{t'=0}^{\infty} \varrho^{t'} \Big[& \mathbf{y}^\top_{t+t'}\, \mathbf{Q}\, \mathbf{y}_{t+t'} + 2\, \mathbf{y}^\top_{t+t'}\, \mathbf{U} \begin{bmatrix} \mathbf{w}_{t+t'} \\ \mathbf{w}^*_{t+t'} \end{bmatrix} \\ & + \mathbf{w}^\top_{t+t'}\, \mathbf{R} \begin{bmatrix} \mathbf{w}_{t+t'} \\ \mathbf{w}^*_{t+t'} \end{bmatrix} \Big] \end{aligned} \tag{B.5}$$

Where we use the definitions $\mathbf{Q} = \mathbf{E}^\top_1\, \Upsilon\, \mathbf{E}_1$, $\mathbf{U} = \mathbf{E}^\top_1\, \Upsilon\, \mathbf{E}_2$, and $\mathbf{R} = \mathbf{E}^\top_2\, \Upsilon\, \mathbf{E}_2$. We also introduce the notation $\mathbf{y}^\top_t = [\mathbf{z}^\top_t, \mathbf{x}^\top_t]$ is the state vector, of dimension $n_s \times 1$, where $n_s = n_p + n_f$. For the foreign government, the welfare loss is

$$U_t = \frac{1}{2} \sum_{t'=0}^{\infty} \varrho^{t'}\, \mathbf{s}^{*\top}_{t+t'}\, \Upsilon^* \mathbf{s}^*_{t+t'} \tag{B.6}$$

which, using B.3 yields an expression isomorph to B.5 that is not reproduced here.

For the vectors that have the dimension $n_s \times 1$, it is convenient to partition the vector into the first n_p element and the n_f elements that follow. Using this notation, for example

$$\mathbf{y}_t \equiv \begin{bmatrix} \mathbf{y}_{p,t} \\ \mathbf{y}_{f,t} \end{bmatrix} \tag{B.7}$$

where here of course $\mathbf{y}_{\mathrm{p,t}} = \mathbf{z}_{\mathrm{t}}$ and $\mathbf{y}_{\mathrm{p}} = \mathbf{z}_{\mathrm{t}}$. It is also inconvenient to introduce a similar notation for matrices. Let $\mathbf{X}$ be any matrix of dimension $\mathrm{n_s} \times \mathrm{n_s}$, then write

$$\mathbf{X} \equiv \begin{bmatrix} \mathbf{X}_{\mathrm{p,p}} & \mathbf{X}_{\mathrm{p,f}} \\ \mathbf{X}_{\mathrm{f,p}} & \mathbf{X}_{\mathrm{f,f}} \end{bmatrix} \tag{B.8}$$

such that $\mathbf{X}_{\mathrm{p,p}}$ is of dimension $\mathrm{n_p} \times \mathrm{n_p}$ $\mathbf{X}_{\mathrm{f,p}}$ is of dimension $\mathrm{n_f} \times \mathrm{n_p}$ $\mathbf{X}_{\mathrm{p,f}}$ is of dimension $\mathrm{n_p} \times \mathrm{n_f}$ and $\mathbf{X}_{\mathrm{f,f}}$ is of dimension $\mathrm{n_f} \times \mathrm{n_f}$. In addition, portion $\mathbf{B}$ into the part that corresponds to the domestic and the foreign instrument

$$\mathbf{B} = \begin{bmatrix} \mathbf{B} \\ \mathbf{B}^{*} \end{bmatrix} \tag{B.9}$$

and adopt the same notation for $\mathbf{E}_2$ and $\mathbf{E}_2^{*}$. We will make repeated use of these notational conventions in the remainder of the appendix, when we develop the solution procedures.

B.1 The cooperative optimal policy with reputation

To compute this equilibrium we assume that $\varrho = \varrho^{*}$. The target to maximise is

$$\begin{aligned} U_{\mathrm{t}}^{\mathrm{c}} = \frac{1}{2} \sum_{\mathrm{t}'=0}^{\infty} \varrho^{\mathrm{t}'} \Bigg\{ & \mathbf{y}_{\mathrm{t+t'}}^{\top} \, \mathbf{Q}^{\mathrm{c}} \, \mathbf{y}_{\mathrm{t+t'}} + 2 \, \mathbf{y}_{\mathrm{t+t'}}^{\top} \, \mathbf{U}^{\mathrm{c}} \begin{bmatrix} \mathbf{w}_{\mathrm{t+t'}} \\ \mathbf{w}_{\mathrm{t+t'}}^{*} \end{bmatrix} \\ & + \mathbf{w}_{\mathrm{t+t'}}^{\top} \, \mathbf{R}^{\mathrm{c}} \begin{bmatrix} \mathbf{w}_{\mathrm{t+t'}} \\ \mathbf{w}_{\mathrm{t+t'}}^{*} \end{bmatrix} \Bigg\} \end{aligned} \tag{B.10}$$

where $\mathbf{Q}^{\mathrm{c}} = \alpha \, \mathbf{Q} + (1-\alpha) \, \mathbf{Q}^{*} \, \mathbf{U}^{\mathrm{c}} = \alpha \, \mathbf{U} + (1-\alpha) \, \mathbf{U}^{*}$, and $\mathbf{R}^{\mathrm{c}} = \alpha \, \mathbf{R} + (1-\alpha) \, \mathbf{R}^{*}$.

By standard theory of Lagrangian multipliers, we then minimise the Lagrangian

$$\mathcal{L}_0 = U_0 + \sum_{\mathrm{t}=0}^{\infty} \varrho^{\mathrm{t}} \boldsymbol{\lambda}_{\mathrm{t'}} \left[\mathbf{A} \, \mathbf{y}_{\mathrm{t}} + \mathbf{B} \, \mathbf{w}_{\mathrm{t}}^{\mathrm{c}} - \mathbf{y}_{\mathrm{t+1}} \right] \tag{B.11}$$

with respect to $\{\mathbf{y}_{\mathrm{t}}\}_{\mathrm{t}=0}^{\infty}$, $\{\boldsymbol{\lambda}_{\mathrm{t}}\}_{\mathrm{t}=0}^{\infty}$, and $\{\mathbf{w}_{\mathrm{t}}^{\mathrm{c}}\}_{\mathrm{t}=0}^{\infty}$, where $\mathbf{w}_{\mathrm{t}}^{\mathrm{c}} \equiv [\mathbf{w}_{\mathrm{t}}, \mathbf{w}_{\mathrm{t}}^{*}]$. This gives the first order conditions that

$$\mathbf{w}_{\mathrm{t}}^{\mathrm{c}} = -\mathbf{R}^{\mathrm{c}\,-1} \left[\varrho \, \mathbf{B}^{\top} \, \boldsymbol{\lambda}_{\mathrm{t+1}} + \mathbf{U}^{\mathrm{c}\,\top} \, \mathbf{y}_{\mathrm{t}} \right] \tag{B.12}$$

$$\mathbf{U}^{\mathrm{c}} \, \mathbf{w}_{\mathrm{t}}^{\mathrm{c}} = \boldsymbol{\lambda}_{\mathrm{t}} - \varrho \, \mathbf{A}^{\top} \, \boldsymbol{\lambda}_{\mathrm{t+1}} - \mathbf{Q}^{\mathrm{c}} \, \mathbf{y}_{\mathrm{t}} \tag{B.13}$$

together with the original constraint

$$\mathbf{y}_{\mathrm{t+1}} = \mathbf{A} \, \mathbf{y}_{\mathrm{t}} + \mathbf{B} \, \mathbf{w}_{\mathrm{t}}^{\mathrm{c}} \tag{B.14}$$

Equations (B.12), (B.13) and (B.14) hold for $\mathrm{t} \geq 1$. They can be written in state-space form as

$$\begin{aligned} & \begin{bmatrix} \mathbf{I} & \varrho \, \mathbf{B} \, \mathbf{R}^{\mathrm{c}\,-1} \, \mathbf{B}^{\top} \\ \mathbf{0} & \varrho \, (\mathbf{A}^{\top} - \mathbf{U}^{\mathrm{c}} \, \mathbf{R}^{\mathrm{c}\,-1} \, \mathbf{B}^{\top}) \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\mathrm{t+1}} \\ \boldsymbol{\lambda}_{\mathrm{t+1}} \end{bmatrix} = \\ & \begin{bmatrix} \mathbf{A} - \mathbf{B} \, \mathbf{R}^{\mathrm{c}\,-1} \, \mathbf{U}^{\mathrm{c}\,\top} & \mathbf{0} \\ -\mathbf{Q}^{\mathrm{c}} + \mathbf{U}^{\mathrm{c}} \, \mathbf{R}^{\mathrm{c}\,-1} \, \mathbf{U}^{\mathrm{c}\,\top} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{\mathrm{t}} \end{bmatrix} \end{aligned} \tag{B.15}$$

The solution to (B.15) requires $2\,\mathrm{n_s}$ boundary conditions. The first order condition in $\mathrm{t}=0$, requires that

$$\boldsymbol{\lambda}_0^\top \,\mathrm{d}\,\mathbf{y}_0 = 0 \tag{B.16}$$

Within $\mathbf{y}_0$ the first $\mathrm{n_p}$ elements are predetermined, therefore $\mathrm{d}\mathbf{y}_0^{\mathrm{p}} = 0$, whereas the $\mathrm{n_f}$ elements that follow are free and therefore require from (B.16) that

$$\boldsymbol{\lambda}_{\mathrm{f},0} = \mathbf{0} \tag{B.17}$$

This gives $\mathrm{n_f}$ boundary condition to solve (B.15). The initial value $\mathbf{z}_0$ gives $\mathrm{n_p}$ more conditions. Finally the transversality condition

$$\lim_{\mathrm{t}\to\infty} \varrho^{\mathrm{t}}\,\boldsymbol{\lambda}_{\mathrm{t}} = \mathbf{0} \tag{B.18}$$

provides $\mathrm{n_s}$ more conditions, which complete to the required $2\,\mathrm{n_s}$ boundary conditions. The solution takes the form

$$\boldsymbol{\lambda}_{\mathrm{t}} = \mathbf{S}^{\mathrm{c}}\,\mathbf{y}_{\mathrm{t}} \tag{B.19}$$

Substituting into (B.13) we get

$$\begin{aligned} \mathbf{w}_{\mathrm{t}}^{\mathrm{c}} &= -\left(\mathbf{R}^{\mathrm{c}} + \mathbf{B}^\top\mathbf{S}^{\mathrm{c}}\,\mathbf{B}\right)^{-1}\left(\mathbf{B}^\top\mathbf{S}^{\mathrm{c}}\,\mathbf{A} + \mathbf{U}^{\mathrm{c}\top}\right)\mathbf{y}_{\mathrm{t}} \\ &= -\mathbf{F}^{\mathrm{c}}\,\mathbf{y}_{\mathrm{t}} \end{aligned} \tag{B.20}$$

say, where $\mathbf{S}^{\mathrm{c}}$ is the solution to the Ricatti matrix equation

$$\begin{aligned} \mathbf{S}^{\mathrm{c}} = {}& \mathbf{Q}^{\mathrm{c}} - \mathbf{U}^{\mathrm{c}}\,\mathbf{F}^{\mathrm{c}} - \mathbf{F}^{\mathrm{c}\top}\,\mathbf{U}^{\mathrm{c}\top} + \mathbf{F}^{\mathrm{c}\top}\,\mathbf{R}^{\mathrm{c}}\,\mathbf{F}^{\mathrm{c}} \\ & + (\mathbf{A} - \mathbf{B}\,\mathbf{F}^{\mathrm{c}})^\top\,\mathbf{S}^{\mathrm{c}}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F}^{\mathrm{c}}) \end{aligned} \tag{B.21}$$

Finally, to complete the solution we express the non-predetermined variables at time t, $\begin{bmatrix}\boldsymbol{\lambda}_{\mathrm{p,t}}^\top & \boldsymbol{\lambda}_{\mathrm{p,t}}^\top\end{bmatrix}^\top$ in terms of the predetermined variables $\begin{bmatrix}\mathbf{z}_{\mathrm{t}}^\top & \boldsymbol{\lambda}_{2,\mathrm{t}}^\top\end{bmatrix}^\top$. Rearranging (B.19), we obtain

$$\begin{aligned} \begin{bmatrix}\boldsymbol{\lambda}_{\mathrm{p,t}} \\ \mathbf{x}_{\mathrm{t}}\end{bmatrix} &= \begin{bmatrix}\mathbf{S}^{\mathrm{c}}{}_{\mathrm{p,p}} - \mathbf{S}^{\mathrm{c}}{}^{-1}_{\mathrm{f,f}}\,\mathbf{S}^{\mathrm{c}}{}_{\mathrm{f,p}} & \mathbf{S}^{\mathrm{c}}{}_{\mathrm{p,f}}\,\mathbf{S}^{\mathrm{c}}{}^{-1}_{\mathrm{f,f}} \\ -\mathbf{S}^{\mathrm{c}}{}^{-1}_{\mathrm{f,f}}\,\mathbf{S}^{\mathrm{c}}{}_{\mathrm{f,p}} & \mathbf{S}^{\mathrm{c}}{}^{-1}_{\mathrm{f,f}}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \\ &= -\mathbf{N}^{\mathrm{c}}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \end{aligned} \tag{B.22}$$

say. Substituting into (B.20) gives

$$\begin{aligned} \mathbf{w}^{\mathrm{c}}{}_{\mathrm{t}} &= -\mathbf{F}^{\mathrm{c}}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ -\mathbf{N}^{\mathrm{c}}{}_{\mathrm{f,p}} & -\mathbf{N}^{\mathrm{c}}{}_{\mathrm{f,f}}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \\ &= \mathbf{G}^{\mathrm{c}}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \end{aligned} \tag{B.23}$$

say, and combining (B.14), (B.20) and (B.22) gives

$$\begin{aligned} \begin{bmatrix}\mathbf{z}_{\mathrm{t}+1} \\ \boldsymbol{\lambda}_{2,\mathrm{t}+1}\end{bmatrix} &= \mathbf{T}^{\mathrm{c}}\,(\mathbf{A} - \mathbf{B}\,\mathbf{F}^{\mathrm{c}})\,\mathbf{T}^{\mathrm{c}\,-1}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \qquad \text{where} \quad \mathbf{T}^{\mathrm{c}} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{S}^{\mathrm{c}}{}_{\mathrm{f,p}} & \mathbf{S}^{\mathrm{c}}{}_{\mathrm{f,f}}\end{bmatrix} \\ &= \mathbf{H}^{\mathrm{c}}\begin{bmatrix}\mathbf{z}_{\mathrm{t}} \\ \boldsymbol{\lambda}_{2,\mathrm{t}}\end{bmatrix} \end{aligned} \tag{B.24}$$

say. Given the solution $\mathbf{S}^c$ to the Ricatti equation (B.21), equations (B.22) to (B.24) completely characterize the solution to the optimization problem. The solution can be expressed as a feedback on the history of the state vectors. At $t = 0$, this feedback is simply given by (B.23), with $t = 0$. To find the feedback for for the following periods, use (B.24) to write

$$\boldsymbol{\lambda}_{2,t+1} = \mathbf{H}^c{}_{f,p}\,\mathbf{z}_t + \mathbf{H}^c{}_{f,f}\,\boldsymbol{\lambda}_{2,t} \tag{B.25}$$

Solving (B.25) and using (B.16), I find

$$\boldsymbol{\lambda}_{2,t+1} = \mathbf{H}^c{}_{f,p}\sum_{t'=1}^{t}(\mathbf{H}^c{}_{f,f})^{t-1}\mathbf{z}_{t-t'} \tag{B.26}$$

Hence the feedback form of the rule $\mathbf{w}_t = \mathbf{G}^c{}_p\,\mathbf{z}_t + \mathbf{G}^c{}_f\,\boldsymbol{\lambda}_{f,t}$ can be expressed solely in terms of the (at time t) predetermined variables $\mathbf{z}_t$.

Finally let us evaluate the welfare loss along the trajectory or "cost-to-go". From the envelope theorem and the first order condition (B.16), we have that

$$\frac{dU}{d\mathbf{y}_0} = \frac{d\mathcal{L}_0}{d\mathbf{y}_0} = \boldsymbol{\lambda}_0^\top \tag{B.27}$$

Hence from (B.19) on integration we have

$$U_0 = \frac{1}{2}\,\mathbf{y}_0^\top\,\mathbf{S}^c\,\mathbf{y}_0 \tag{B.28}$$

at time $t' = 0$. At time t this becomes

$$U_t = \frac{1}{2}\,\mathbf{y}_t^\top\,\mathbf{S}^c\,\mathbf{y}_t \tag{B.29}$$

Another way of expressing U_t, which will proof useful, is found by eliminating $\mathbf{x}_t$ in (B.29) using (B.22). We obtain

$$U_t = -\frac{1}{2}\left[\mathrm{trace}(\mathbf{N}^c_{p,p}\,\mathbf{z}_t\,\mathbf{z}_t^\top) + \mathrm{trace}(\mathbf{N}^c_{f,f}\,\boldsymbol{\lambda}_{2,t}\,\boldsymbol{\lambda}_{2,t}^\top)\right] \tag{B.30}$$

which at $t = 0$, using (B.17) becomes

$$U_0 = -\frac{1}{2}\left[\mathrm{trace}(\mathbf{N}^c_{p,p}\,\mathbf{z}_0\,\mathbf{z}_0^\top)\right] \tag{B.31}$$

B.2 The TC regime of cooperation without reputation

The solution here is very similar to the one developed in Subsection A.3. To sum up, we develop an iterative scheme

$$\begin{aligned}
\mathbf{J}^c_t &= -(\mathbf{A}_{f,f} + \mathbf{N}^c_{t+1}\,\mathbf{A}_{f,p})^{-1}(\mathbf{N}^c_{t+1}\,\mathbf{A}_{p,p} + \mathbf{A}_{f,p}) \\
\mathbf{K}^c_t &= -(\mathbf{A}_{f,f} + \mathbf{N}^c_{t+1}\,\mathbf{A}_{f,p})^{-1}\,(\mathbf{N}^c_{t+1}\,\mathbf{B}_p + \mathbf{B}_f) \\
\mathbf{N}^c_t &= -\mathbf{J}^c_t + \mathbf{K}^c_t\,\mathbf{G}^c_t \\
\mathbf{G}^c_t &= (\bar{\mathbf{R}}^c_t + \varrho\,\bar{\mathbf{B}}_t^\top\,\mathbf{S}_{t+1}\,\bar{\mathbf{B}}_t)\,(\bar{\mathbf{U}}^c_t + \varrho\,\bar{\mathbf{B}}_t^\top\,\mathbf{S}_{t+1}\,\bar{\mathbf{A}}_t) \\
\bar{\mathbf{Q}}^c_t &= \mathbf{Q}^c_{p,p} + \mathbf{J}^{c\top}_t\,\mathbf{Q}^c_{f,p} + \mathbf{Q}^c_{p,f}\,\mathbf{J}^c_t + \mathbf{J}^{c\top}_t\,\mathbf{Q}^c_{f,f}\,\mathbf{J}^c_t \\
\bar{\mathbf{U}}^c_t &= \mathbf{U}^c_p + \mathbf{Q}^c_{p,f}\,\mathbf{K}^c_t + \mathbf{J}^{c\top}_t\,\mathbf{U}^c_f + \mathbf{J}^{c\top}_t\,\mathbf{Q}^c_{f,f}\,\mathbf{J}^c_t \\
\bar{\mathbf{R}}^c_t &= \mathbf{R} + \mathbf{U}^{c\top}_f\,\mathbf{K}^c_t + \mathbf{K}^{c\top}_t\,\mathbf{U}^c_f + \mathbf{K}^{c\top}_t\,\mathbf{Q}^c_{f,f}\,\mathbf{K}^c_t \\
\mathbf{S}_t &= \bar{\mathbf{Q}}^c_t + \bar{\mathbf{U}}^c_t\,\mathbf{G}^c_t + \mathbf{G}^{c\top}_t\,\mathbf{U}^{c\top}_t + \mathbf{G}^{c\top}_t\,\bar{\mathbf{R}}^c_t\,\mathbf{G}^c_t \\
&\quad + \varrho\,(\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}^c_t)^\top\,\mathbf{S}_{t+1}(\bar{\mathbf{A}}_t + \bar{\mathbf{B}}_t\,\mathbf{G}^c_t)
\end{aligned} \tag{B.32}$$

If the system converges, then the solution is given by

$$\mathbf{w}_t = \mathbf{F}^c\,\mathbf{z}_t \tag{B.33}$$

$$\mathbf{x}_t = \mathbf{N}^c\,\mathbf{z}_t \tag{B.34}$$

where

$$\mathbf{z}_{t+1} = [\mathbf{A}_{p,p} + \mathbf{A}_{p,f}\,\mathbf{J}^c - (\mathbf{B}_p + \mathbf{A}_{p,f}\,\mathbf{K}^c)\,\mathbf{F}^c]\,\mathbf{z}_t \tag{B.35}$$

B.3 The TN regime of non-cooperation without reputation

The TC regime is a Nash equilibrium. It is found by iterating the sequence (B.32) between the players, each one in turn updating their decisions and take the others' as given. ACES does that by stating with the domestic policy maker, then the private sector, then the foreign policy maker,then the private sector again, before returning to the domestic policy maker. Little is known about the convergence properties of this algorithm but it turns out to be quite robust.